Mathematical Methods for the Design of Gravity Thrust Space Trajectories

By

Michael Andrew Minovitch

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Approved:

S. Kobayashi ....

R. K. Sachs ....

George J. Weibel ....

Committee in Charge

....

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Abstract

It is possible to propel a free-fall space vehicle from planet to planet to any place in the entire solar system by utilizing the gravitational perturbation of each planet passed as a powerful vehicle thrust source. Except for guidance, these trajectories do not require any rocket propulsion after the vehicle leaves the vicinity of the launch planet. Consequently, since these thrust forces increase in direct proportion to vehicle mass, as described by the equivalence principle, it does not matter how massive the vehicle is once it is launched. This concept of vehicle propulsion has had a significant effect in the planning of future interplanetary space missions. We shall call these trajectories "Gravity Thrust Space Trajectories".

This dissertation presents a collection of mathematical techniques which I have developed in the course of studying Gravity Thrust trajectories. A strict mathematical treatment would involve the unsolved N-body problem of analytical mechanics. This difficulty is circumvented by assuming that, at any given instant, the trajectory is strictly

Keplerian with respect to either a nearby planet or with respect to the Sun. The techniques are essentially vector in character and are ideally suited to studying Gravity Thrust trajectories in a three-dimensional space. Much use is made of Lambert's equations for Keplerian motion. The ambiguities usually inherent in the elliptical forms of these equations are completely removed. Solutions to problems corresponding to specific missions such as multiple planetary encounters, solar-probe, and out-of-ecliptic trajectories are given. Although this work is intended to be purely theoretical, relevant practical aspects will also be discussed.

THIS DISSERTATION IS DEDICATED TO:

John Fitzgerald Kennedy

35th President of the United States

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Introduction

The central subject of this dissertation involves planetary perturbations and, in particular, how they effect the motion of free-fall interplanetary space vehicles. Before going into the mathematical details of the paper, it is interesting to review the great lengths to which some researchers have gone to compensate for the perturbation factor, which has, ironically, turned out to be the key to many interplanetary missions previously thought to require extremely powerful launch vehicles.

If a comet or free-fall interplanetary space vehicle moving under the gravitational influence of the Sun remains sufficiently far from any of the various planets, its path is essentially that of a conic section. These Keplerian orbits are well known solutions of the two-body problem. On the other hand, if the object passes close by a planet, the resulting gravitational perturbation usually becomes non-negligible, and, thus, the orbit's constant Keplerian nature relative to the Sun will be destroyed. The early pioneers in astrodynamics tended to view these planetary perturbations as annoying disturbances of their purely Keplerian orbits. For example, Hohmann (1), while considering the flight dynamics of interplanetary reconnaissance vehicles, discovered an extremely rare elliptical trajectory which would (if it were not for the perturbations) allow a vehicle to intersect the orbits of both Venus and Mars respectively just as these planets passed by, and return to Earth. To make his trajectory realizable he simply cancelled out the perturbations by applying rocket thrust equal to the perturbating force but of opposite direction. Unfortunately, Hohmann had to use up a total Δv of 4.1 km/sec to cancel out these perturbations. Later, in 1956, Crocco (2) devised a more sophisticated method of cancelling out these perturbations. He discovered an equally remarkable constant elliptical path which would also pass the orbits of Venus and Mars just as these planets appeared, but in reverse order. To this ideal undisturbed path he added the resulting perturbation encountered by passing mars at a short distance. The effect of this perturbation was calculated and compared to the undisturbed path. Finally, the perturbation due to passing Venus at various distances was introduced and examined. By varying the Venus passing distance by the right amount, he was able to cancel out the effect of the Mars perturbation and obtain a final trajectory very close to the original undisturbed path. More recent trajectory analysts (3) often avoided considering perturbations altogether by simply not allowing a free-fall reconnaissance vehicle to pass very close to a perturbing planet.

When these classical trajectory design procedures were used to obtain numerical orbit determinations via modern high speed digital computers, the results only confirmed the conclusions reached earlier by the crude hand computations of Hohmann (1) and Oberth (4). Except for relatively simple fly-by missions to Mars or Venus, it was concluded that most interplanetary missions would require extremely high launch energies and/or very long flight times. The dual planet fly-by trajectories of Hohmann and Crocco were found to require so much energy that some analysts viewed them as "interesting academic pastimes" (Ch. 5 ref. 3). Nevertheless, the early 1960's was an era of enthusiasm for space travel, and rough designs for the required super boosters (known variously as Novas, Super-Novas and Sea Dragons) were drawn (5).

Of the many aspects involved in the planning of interplanetary space missions, trajectory design is the most important. Trajectories determine launch velocities, and launch velocities, together with payload mass, determine required launch energies and launch vehicle size. In order to minimize the amount of rocket thrust which would otherwise be required for some interplanetary missions, I proposed a method of trajectory design (6) which was based primarily upon the perturbational forces a nearby planet can impart to a passing space vehicle. This method offered the interesting theoretical possibility of free and unlimited space travel anywhere in the entire solar system by simply bouncing the vehicle off the back sides of various moving regions of perturbation which, conveniently enough, encompass planets. Except for maintaining proper guidance, no on-board rocket thrust is required once the vehicle is launched. The thrust required for interplanetary mobility is obtained via precisely calculated gravitational interactions with passing planets. Control is exercised by selecting pre-calculated planetary approach trajectories. Since these perturbational thrust forces increase in direct proportion to vehicle mass, they can be viewed as rocket engines with almost unlimited power. Once the vehicle is launched, it will not matter if its mass is 10^3 kgm or 10^9 kgm. An engine of this type keeps the entire Earth in its orbit around the Sun. These engines are "clean burning" and do not add to vehicle invironmental pollution.

The quantitative study of dynamics in a three-dimensional space is most conveniently made with the use of vector analysis. Hence, a convenient vector method will be developed for defining Keplerian orbits. With these analytical techniques, no assumptions regarding the geometry of the solar system will be necessary. It will not matter how eccentric a planet's orbit is or how much its orbital plane is inclined to the ecliptic.

Chapter I

Mathematical Preliminaries

1.1 A Vector Treatment of the General Two Body Problem

We begin by writing down the equations of motion for two bodies of mass m_1 and m_2 moving under their mutual gravitational influence. These equations are obtained by equating the individual dynamical forces to the gravitational forces.

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -Gm_1m_2\frac{\vec{r}_1 - \vec{r}_2}{r_{12}^3} \qquad (1.1.1)$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = -Gm_2m_1\frac{\vec{r}_2 - \vec{r}_1}{r_{12}^3} . \qquad (1.1.2)$$

The vectors $\vec{r}_1$ and $\vec{r}_2$ denote the object's position vectors relative to some arbitrary inertial coordinate system Σ . The quantity r_{12} is defined as the distance between m_1 and m_2 (which we assume to be spherically symmetric) and G is the universal gravitational constant. Let M denote the total mass of the system and $\vec{P}$ the center of mass. Hence it follows that

$$\begin{aligned} M &= m_1 + m_2 \\ M\vec{P} &= m_1\vec{r}_1 + m_2\vec{r}_2 . \end{aligned} \qquad (1.1.3)$$

Define new vectors $\vec{R}_1$ and $\vec{R}_2$ by

$$\vec{R}_1 = \vec{r}_1 - \vec{P} \qquad \vec{R}_2 = \vec{r}_2 - \vec{P} . \qquad (1.1.4)$$

By substituting these equations into (1.1.3) we obtain

$$M\vec{P} = m_1(\vec{R}_1+\vec{P}) + m_2(\vec{R}_2+\vec{P}) = M\vec{P}+m_1\vec{R}_1 + m_2\vec{R}_2 .$$

Hence it follows that

$$m_1\vec{R}_1 + m_2\vec{R}_2 = 0 . \qquad (1.1.5)$$

Moreover, by differentiating with respect to time we obtain

$$m_1\frac{d\vec{R}_1}{dt} + m_2\frac{d\vec{R}_2}{dt} = 0$$

and, by denoting the time derivative of $\vec{R}_i$ by $\vec{V}_i$ $(i = 1,2)$, this equation becomes

$$m_1\vec{V}_1 + m_2\vec{V}_2 = 0 . \qquad (1.1.6)$$

From equation (1.1.3) it follows that

$$M \frac{d^2\vec{P}}{dt} = m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} .$$

Hence by making use of the equations of motion (1.1.1) and (1.1.2) we obtain

$$M\frac{d^2\vec{P}}{dt^2} = \frac{-Gm_1m_2}{r_{12}^3}(\vec{r}_1-\vec{r}_2+\vec{r}_2-\vec{r}_1) = 0.$$

Therefore

$$\frac{d\vec{P}}{dt} = \vec{V}_0$$

where $\vec{V}_0$ is a constant. Now since equations (1.1.1) and (1.1.2) are invariant under Galilean transformations $\vec{r} \to \vec{r} + \vec{V}_0 t$ we may assume $\vec{V}_0 = 0$ and $\vec{P} = 0$. We now differentiate the vector $\vec{R}_i \times \vec{V}_i$ with respect to time t.

$$\frac{d}{dt}(\vec{R}_i \times \vec{V}_i) = \vec{V}_i \times \vec{V}_i + \vec{R}_i \times \frac{d\vec{V}_i}{dt} .$$

Hence by making use of (1.1.1) (or (1.1.2) and noting that $\vec{V}_i \times \vec{V}_i = 0$ we obtain

$$\frac{d}{dt}(\vec{R}_i \times \vec{V}_i) = \vec{R}_i \times [-Gm_j \frac{(\vec{R}_i - \vec{R}_j)}{R_{12}^3}] .$$

In view of (1.1.5) it follows that

$$\frac{d}{dt}(\vec{R}_i \times \vec{V}_i) = 0 \qquad (i = 1,2; \quad j = 1,2; \quad i \neq j) .$$

Consequently, the vectors $\vec{R}_i \times \vec{V}_i$ are constant and shall be denoted by $\lambda_i \vec{h}_i$ where λ_i is some constant scalar to be determined later. Hence we have

$$\lambda_i \vec{h}_i = \vec{R}_i \times \vec{V}_i . \tag{1.1.7}$$

These vectors are proportional to the angular momentum vectors of m_i $(i = 1,2)$. From equations (1.1.5) and (1.1.6) it follows that

$$\lambda_i \vec{h}_i = (-\frac{m_j}{m_i}\vec{R}_j) \times (-\frac{m_j}{m_i}\vec{V}_j) = (\frac{m_j}{m_i})^2 \lambda_j \vec{h}_j . \tag{1.1.8}$$

Throughout this paper we shall denote unit vectors by placing the symbol ^ over the quantity instead of the usual →. Hence the unit vector corresponding to $\vec{R}_i$ will be denoted by

$$\hat{R}_i = \frac{\vec{R}_i}{R_i} .$$

We may therefore express $\lambda_i \vec{h}_i$ as

$$\lambda_i \vec{h}_i = R_i \hat{R}_i \times (R_i \frac{d\hat{R}_i}{dt} + \frac{dR_i}{dt} \hat{R}_i) = R^2 \hat{R}_i \times \frac{d\hat{R}_i}{dt} .$$

Hence, by making use of the equations of motion a third time we obtain

$$\lambda_i \frac{d}{dt}(\vec{V}_i \times \vec{h}_i) = \lambda_i \frac{d\vec{V}_i}{dt} \times \vec{h}_i$$

$$= -Gm_j (\frac{\vec{R}_i - \vec{R}_j}{R_{12}^3}) \times R_i^2 (\hat{R}_i \times \frac{d\hat{R}_i}{dt})$$

$$= -\frac{Gm_j R_i^2}{R_{12}^3} [\vec{R}_i \times (\hat{R}_i \times \frac{d\hat{R}_i}{dt}) - \vec{R}_j \times (\hat{R}_i \times \frac{d\hat{R}_i}{dt})]$$

$$= -\frac{Gm_j R_i^2}{R_{12}^3} [(\vec{R}_i \cdot \frac{d\hat{R}_i}{dt})\hat{R}_i - (\vec{R}_i \cdot \hat{R}_i)\frac{d\hat{R}_i}{dt}$$

$$- (\vec{R}_j \cdot \frac{d\hat{R}_i}{dt})\hat{R}_i + (\vec{R}_j \cdot \hat{R}_i)\frac{d\hat{R}_i}{dt}] .$$

However, since we have

$$\hat{R}_i \cdot \hat{R}_i = 1$$

$$\frac{d}{dt}(\hat{R}_i \cdot \hat{R}_i) = 0 = \frac{d\hat{R}_i}{dt} \cdot \hat{R}_i + \hat{R}_i \cdot \frac{d\hat{R}_i}{dt}$$

which, in view of (1.1.5), implies $\vec{R}_k \cdot \frac{d\hat{R}_i}{dt} = 0 \quad (i,k = 1,2)$. Consequently, we obtain

$$\lambda_i \frac{d}{dt}(\vec{V}_i \times \vec{h}_i) = \frac{-Gm_j R_i^2}{R_{12}^3}[-R_i - (\frac{m_i}{m_j}R_i)]\frac{d\hat{R}_i}{dt} .$$

However, since $R_{12} = R_1 + R_2$ we may write, according to (1.1.5)

$$R_{12} = (1 + \frac{m_i}{m_j})R_i .$$

Therefore,

$$\lambda_i \frac{d}{dt}(\vec{V}_i \times \vec{h}_i) = \frac{Gm_j R_i^3 (1 + \frac{m_i}{m_j})\frac{d\hat{R}_i}{dt}}{R_i^3(1 + \frac{m_i}{m_j})^3}$$

$$= Gm_j (\frac{m_j}{M})^2 \frac{d\hat{R}_i}{dt} .$$

By defining a constant scalar μ_i by

$$\mu_i = Gm_j (\frac{m_j}{M})^2 \qquad (1.1.9)$$

we obtain

$$\lambda_i \frac{d}{dt}(\vec{V}_i \times \vec{h}_i) = \mu_i \frac{d\hat{R}_i}{dt} .$$

An integration of this equation yields

$$\lambda_i \vec{V}_i \times \vec{h}_i = \mu_i (\hat{R}_i + \vec{e}_i) \qquad (1.1.10)$$

where the constant vector of integration is denoted by $\mu_i \vec{e}_i$. It is easy to show that equations (1.1.5), (1.1.6) and (1.1.8) imply

$$\vec{e}_i = -\vec{e}_j . \qquad (i \neq j) \tag{1.1.11}$$

The vectors $\vec{h}_i$ and $\vec{e}_i$ are clearly orthogonal to each other and hence represent only five arbitrary constants of integration. Consequently, the four vectors $\vec{h}_1$, $\vec{e}_1$ and $\vec{h}_2$, $\vec{e}_2$ represent ten of the twelve constants of integration corresponding to the two second order vector differential equations (1.1.1) and (1.1.2). The last two constants of integration will correspond to the time of perifocal passage t_p.

Now in view of the fact that the vectors $\vec{h}_1$ and $\vec{h}_2$ are constant, equations (1.1.7) and (1.1.8) imply that the motion of m_1 and m_2 is confined to a fixed plane perpendicular to the vector $\hat{h}_1 = \hat{h}_2 = \hat{h}$. We define the angle θ_i (which is called true anomaly) by

$$\theta_i = \measuredangle\, \vec{e}_i, \vec{R}_i \qquad (i = 1,2)$$

measured in the direction of motion from $\vec{e}_i$ to $\vec{R}_i$. From (1.1.5) and (1.1.11) it follows that $\theta_1 = \theta_2$. We shall denote this angle by θ. Hence from (1.1.7) and (1.1.10) we obtain

$$\lambda_i^2 h_i^2 = \lambda_i \vec{h}_i \cdot (\vec{R}_i \times \vec{V}_i) = \lambda_i \vec{R}_i \cdot (\vec{V}_i \times \vec{h}_i) = \mu_i \vec{R}_i \cdot (\hat{R}_i + \vec{e}_i)$$

Consequently

$$R_i = \frac{\dfrac{\lambda_i^2 h_i^2}{\mu_i}}{1 + \vec{e}_i \cdot \hat{R}_i} \tag{1.1.12}$$

which in view of (1.1.5) and (1.1.11) can be expressed as

$$R_i = \frac{\dfrac{\lambda_i^2 h_i^2}{\mu_i}}{1 + e \cos\theta} \tag{1.1.13}$$

where $e = |\vec{e}_1| = |\vec{e}_2|$. This is the general equation of a conic section in polar form with eccentricity e and semi-latus rectum ℓ_i given by

$$\ell_i = \frac{\lambda_i^2 h_i^2}{\mu_i} \tag{1.1.14}$$

Hence both bodies move along conic paths of equal eccentricity and semi-latus rectum ℓ_i given by (1.1.14). Now by (1.1.9) it follows that

$$\mu_2 = Gm_1\left(\frac{m_1}{M}\right)^2 = Gm_2\left(\frac{m_2}{M}\right)^2\left(\frac{m_1}{m_2}\right)^3 = \mu_1\left(\frac{m_1}{m_2}\right)^3 .$$

Hence by (1.1.8) we obtain

$$\ell_1 = \frac{\lambda_1^2 h_1^2}{\mu_1} = \frac{\left(\dfrac{m_2}{m_1}\right)^4 \lambda_2^2 h_2^2}{\left(\dfrac{m_2}{m_1}\right)^3 \mu_2} = \left(\frac{m_2}{m_1}\right) \ell_2 .$$

The relation between ℓ_i and a_i where a_i denotes the semi-major axis of the i^{th} orbit is

$$\ell_i = a_i\ |1 - e_i^2| . \tag{1.1.15}$$

Hence

$$a_1 = \left(\frac{m_2}{m_1}\right) a_2 .$$

By making use of (1.1.10) we may write

$$\lambda_i \vec{h}_i \times (\vec{V}_i \times \vec{h}_i) = \mu_i \vec{h}_i \times (\hat{R}_i + \vec{e}_i) = \lambda_i [(\vec{h}_i \cdot \vec{h}_i)\vec{V}_i - (\vec{h}_i \cdot \vec{V}_i)\vec{h}_i] \ .$$

But, since $\vec{h}_i$ is orthogonal to both $\vec{R}_i$ and $\vec{V}_i$, we obtain

$$\vec{V}_i = \frac{\mu_i}{\lambda_i h_i^2} \vec{h}_i \times (\hat{R}_i + \vec{e}_i)$$

By defining λ_i as

$$\lambda_i = \frac{\mu_i}{h_i^2} \ ,$$

we obtain one of our most important equations

$$\vec{V}_i = \vec{h}_i \times (\vec{e}_i + \hat{R}_i) \ . \tag{1.1.16}$$

We sum up these results by the following theorem:

The Orbit Representation Theorem. Any body moving under the gravitational influence of a second body will describe a conic path with respect to Σ whose geometry is completely defined by two constant and mutually orthogonal vectors $\vec{h}_i$ and $\vec{e}_i$. The velocity vector $\vec{V}_i$ corresponding to any point $\vec{R}_i$ on the orbit is given by (1.1.16).

Suppose m_1 is the mass of the Sun or a particular planet and m_2 is the mass of a free-fall interplanetary space vehicle. Then $m_1 \gg m_2$ and we may set the ratio $m_2/m_1 = 0$. In this case $\vec{R}_1 = 0$ and $\vec{V}_1 = 0$. The subscript 2 may therefore be omitted from $\vec{h}_2$ and $\vec{e}_2$ without confusion. Hence we may express (1.1.7) and (1.1.16) as

$$\vec{h} = \frac{\vec{R} \times \vec{V}}{\ell} \tag{1.1.17}$$

$$\vec{V} = \vec{h} \times (\hat{R} + \vec{e}) \tag{1.1.18}$$

respectively, where

$$\ell = \frac{\mu}{h^2} \tag{1.1.19}$$

and

$$\mu = Gm_1 \; . \tag{1.1.20}$$

We may also express (1.1.13) and (1.1.15) as

$$R = \frac{\ell}{1 + \vec{e} \cdot \hat{R}} \tag{1.1.21}$$

and

$$\ell = a \left| 1 - e^2 \right| \tag{1.1.22}$$

respectively.

Let us now dot multiply each side of equation (1.1.18) by $\vec{V}$. Then, making use of (1.1.19), we obtain

$$V^2 = [\vec{V} \cdot (\vec{h} \times \vec{e}) + \vec{V} \cdot (\vec{h} \times \hat{R})]$$

$$= \frac{1}{R} [R\vec{e} \cdot (\vec{V} \times \vec{h}) + \vec{h} \cdot (\vec{R} \times \vec{V})] \; .$$

In view of equations (1.1.10) and (1.1.17), we may write

$$V^2 = \frac{\mu}{h^2 R} [\frac{\mu}{\ell^2} R\vec{e} \cdot (\vec{e} + \hat{R}) + h^2]$$

By making use of (1.1.21) and (1.1.22), this equation becomes

$$V^2 = \mu [\frac{2}{R} \mp \frac{1}{a}] \; , \tag{1.1.23}$$

where the negative and positive signs correspond to elliptical and hyperbolic orbits respectively. This equation, which is well known in Celestial Mechanics, will be called the "orbital energy equation".

In observational astronomy a heliocentric orbit is usually defined by the so-called classical orbital elements. They are expressed by a, e, i, Ω, ω and t_p and defined

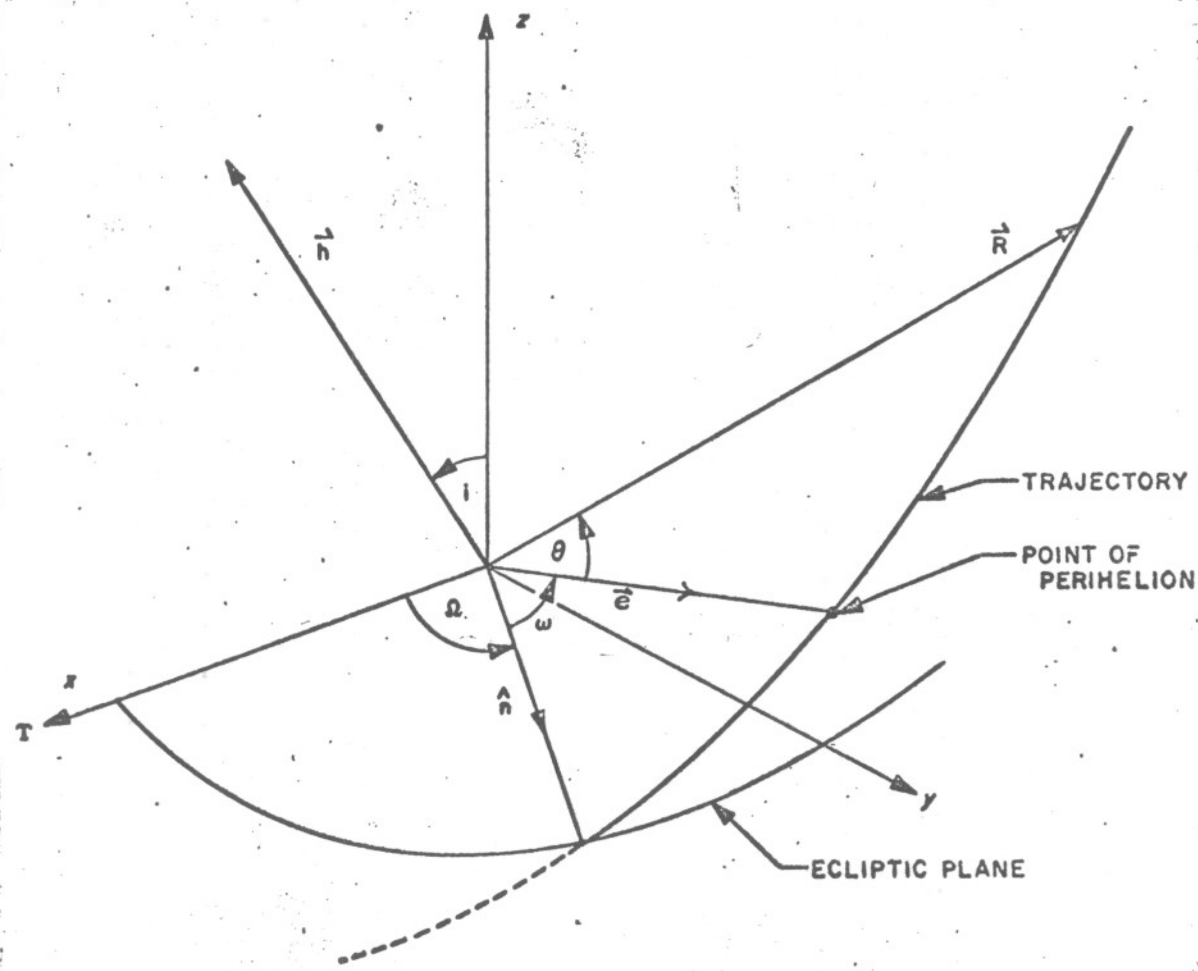


Figure 1. A Geometric Description of the Classical Orbital Elements

by figure 1 where Σ is taken as an elliptic coordinate system. By denoting the component forms of $\vec{e}$ and $\vec{h}$ by

$$\vec{e} = (e_1, e_2, e_3) \qquad \vec{h} = (h_1, h_2, h_3)$$

it is easy to obtain the following transformation equations:

$$\begin{aligned} \cos i &= \frac{h_3}{h} & \sin \Omega &= \frac{h_1}{h \sin i} \\ \cos \omega &= \frac{\hat{n} \cdot \vec{e}}{e} & \cos \Omega &= \frac{-h_2}{h \sin i} \\ e &= (\vec{e} \cdot \vec{e})^{1/2} & a &= \frac{\mu}{(h^2 |1 - e^2|)} \end{aligned} \tag{1.1.24}$$

where $\hat{n} = (\cos \Omega, \sin \Omega, 0)$. If the classical orbital elements are given, the orbital vectors $\vec{e}$ and $\vec{h}$ can be obtained by

$$\begin{aligned} e_1 &= e(\cos \Omega \cos \omega - \cos i \sin \Omega \sin \omega) \\ e_2 &= e(\sin \Omega \cos \omega + \cos i \cos \Omega \sin \omega) \\ e_3 &= e \sin \omega \sin i \\ h_1 &= h \sin \Omega \sin i \\ h_2 &= -h \cos \Omega \sin i \\ h_3 &= h \cos i \end{aligned} \tag{1.1.25}$$

where

$$h = \left[\frac{\mu}{a |1 - e^2|} \right]^{1/2} \tag{1.1.26}$$

The names of these classical orbital elements are as follows:

a = semi-major axis

e = eccentricity

i = inclination

Ω = longitude of the ascending node

ω = argument of perihelion

t_p = time of perihelion passage

Most problems in the new field of astrodynamics are quite different from those of Celestial Mechanics. For example, velocities, which are of prime importance in astrodynamics, are hardly ever used by astronomers in their orbit determinations of comets. Hence, in view of (1.1.18), we shall adopt the vector representation for defining orbits.

1.2 Lambert's Equations

In the above formulation we have seen how the shape and orientation of an arbitrary conic orbit can be defined in a three-dimensional space Σ by two mutually orthogonal orbital vectors $\vec{e}$ and $\vec{h}$. It remains, therefore, to relate the position vector $\vec{R}$ of an object, moving on a conic orbit, with time. In celestial mechanics this is almost always done by Kepler's equations. These equations provide a relationship between a time interval $t_j - t_i > 0$ and the orbit's semi-major axis a , eccentricity e and true anomalies $\theta_i = \theta(t_i)$ and $\theta_j = \theta(t_j)$. It is possible, however, to express this time interval completely independent of the orbit's eccentricity. These equations are called Lambert's equations, named in honor of Johann H. Lambert (1728-1777) who discovered them in 1761. They are generalizations of an equation discovered earlier by Euler for the special case of parabolic orbits. Unfortunately, the generalized equations of Lambert suffer from serious ambiguities and are rarely used in celestial mechanics. We shall discover, however, that by a suitable formulation of these equations, it will be possible to completely eliminate these ambiguities.

Let us define the angle θ_{ij} by

$$\theta_{ij} = \sphericalangle\, \vec{R}_i, \vec{R}_j$$

which will be measured in the plane of motion from $\vec{R}_i = \vec{R}(t_i)$ to $\vec{R}_j = \vec{R}(t_j)$ in the direction of motion where $t_i < t_j$.

Consider the case where $0 < \theta_{ij} < 180^\circ$. In this case the elliptical form of Lambert's equations can be expressed by

$$t_j - t_i = \eta[\varepsilon - \sin \varepsilon) - (\delta - \sin \delta)] \qquad (1.2.1)$$

where $\eta = \sqrt{\frac{a^3}{\mu}}$. The variables ε and δ are related to the vectors $\vec{R}_i$ and $\vec{R}_j$ by the equations

$$R_i + R_j + d_{ij} = 4a \sin^2 \frac{\varepsilon}{2}$$

$$R_i + R_j - d_{ij} = 4a \sin^2 \frac{\delta}{2}$$

where $d_{ij} = |\vec{R}_j - \vec{R}_i|$. Unfortunately, the angles ε and δ are not uniquely determined when the vectors $\vec{R}_i$ and $\vec{R}_j$ are given.

Let $\mathfrak{R}$ represent the region bounded by the arc $\widehat{\vec{R}_i \vec{R}_j}$ and the vectors $\vec{R}_i$ and $\vec{R}_j$. Suppose $\mathfrak{R}$ does not contain the vacant focus. Then by defining

$$s_{ij} = (R_i + R_j + d_{ij})/2$$

$$x_{ij} = 1 - (s_{ij}/a)$$

$$y_{ij} = 1 - (s_{ij} - d_{ij})/a$$

equation (1.2.1) can be expressed as

$$t_j - t_i = \eta[\sqrt{1 - y_{ij}^2} + \sin^{-1} y_{ij} - \sqrt{1 - x_{ij}^2} - \sin^{-1} x_{ij}]. \qquad (1.2.2)$$

On the other hand, if $\mathfrak{R}$ does contain the vacant focus, (1.2.1) becomes

$$t_j - t_i = \eta[\pi + \sqrt{1 - y_{ij}^2} + \sin^{-1} y_{ij} + \sqrt{1 - x_{ij}^2} + \sin^{-1} x_{ij}]. \qquad (1.2.3)$$

Both equations (1.2.2) and (1.2.3) correspond to the case where $\theta_{ij} < 180°$. Their differences demonstrate the serious ambiguities in the elliptical form of Lambert's equation since, in general, one has no way of knowing if $\mathcal{R}$ does or does not contain the vacant focus. Similar ambiguities exist when $180° \leq \theta_{ij} < 360°$. In this case, if $\mathcal{R}$ does not contain the vacant focus

$$t_j - t_i = \eta[2\pi - \sqrt{1 - y_{ij}^2} - \sin^{-1} y_{ij} + \sqrt{i - x_{ij}^2} + \sin^{-1} x_{ij}] \qquad (1.2.4)$$

and if $\mathcal{R}$ does contain the vacant focus

$$t_j - t_i = \eta[\pi - \sqrt{1 - y_{ij}^2} - \sin^{-1} y_{ij} - \sqrt{1 - x_{ij}^2} - \sin^{-1} x_{ij}] . \qquad (1.2.5)$$

Hence, corresponding to one unique Kepler equation there corresponds four different Lambert equations (1.2.2), (1.2.3), (1.2.4) and (1.2.5). It is easy to understand therefore why Lambert's equations are rarely used in practice.

The ambiguities in the hyperbolic and parabolic form of Lambert's equations are due only to the two cases $0 < \theta_{ij} < 180°$ and $180° \leq \theta_{ij} < 360°$. In the first case, the hyperbolic equation becomes

$$t_j - t_i = \eta[\sqrt{x_{ij}^2 - 1} - \cosh^{-1} x_{ij} - \sqrt{y_{ij}^2 - 1} + \cosh^{-1} y_{ij}] \qquad (1.2.6)$$

where x_{ij} and y_{ij} are now defined by

$$x_{ij} = 1 + (s_{ij}/a) \qquad y_{ij} = 1 + (s_{ij} - d_{ij})/a .$$

If $180^\circ \leq \theta_{ij} < 360^\circ$ the equation becomes

$$t_j - t_i = \eta[\sqrt{x_{ij}^2 - 1} - \cosh^{-1} x_{ij} + \sqrt{y_{ij}^2 - 1} + \cosh^{-1} y_{ij}]. \quad (1.2.7)$$

For the special case of parabolic orbits Lambert's equation becomes (assuming $0 < \theta_{ij} < 180^\circ$)

$$t_j - t_i = \frac{1}{3}\sqrt{\frac{2}{\mu}}\,[s_{ij}^{3/2} - (s_{ij} - d_{ij})^{3/2}] \quad (1.2.8)$$

while, if $180^\circ \leq \theta_{ij} < 360^\circ$, the equation becomes

$$t_j - t_i = \frac{1}{3}\sqrt{\frac{2}{\mu}}\,[s_{ij}^{3/2} + (s_{ij} - d_{ij})^{3/2}] \; . \quad (1.2.9)$$

Equations (1.2.8) and (1.2.9) are the Euler equations.

Our main purpose for obtaining the above representations of Lambert's equations lies in the fact that they are more closely related to the points $\vec{R}_i$ and $\vec{R}_j$. The primary independent variables will be t_i, t_j, $\vec{R}_i$, and $\vec{R}_j$. The quantity to be determined will be a . It is clear, however, that in order to determine a , one must know before hand which equation to use. (A derivation of the Lambert function (1.2.1) and its various elliptical and hyperbolic variants can be found in [7]. A discussion concerning the ambiguities is also given.)

1.3 Removing the Ambiguities from Lambert's Equations.

Let F^* and F denote the vacant and occupied foci respectively for any arbitrary elliptical orbit. Let the points $\vec{R}_i$ and $\vec{R}_j$ be denoted by P and Q respectively. Then

$$\overline{FP} = R_i \qquad \overline{FQ} = R_j \; .$$

Hence, by the definition of an ellipse we may write

$$\overline{FP} + \overline{F^*P} = 2a \qquad \overline{FQ} + \overline{F^*Q} = 2a \; .$$

Consequently,

$$s_{ij} = \frac{1}{2}(\overline{FP} + \overline{FQ} + d_{ij}) = \frac{1}{2}(4a - \overline{F^*P} - \overline{F^*Q} + d_{ij}) \; .$$

Since the sum of any two sides of a plane triangle in Euclidean space is greater than or equal to the third side it follows that

$$s_{ij} \leq 2a \; . \qquad (1.3.1)$$

Let us now assume that $\vec{R}_i$ and $\vec{R}_j$ are fixed vectors with $\theta_{ij} < 180^\circ$ and view equations (1.2.2) and (1.2.3) as functions of the single variable a. By substituting $a = s_{ij}/2$ into (1.2.2) and (1.2.3) we obtain identical expressions for the time interval $t_j - t_i$ which we shall denote by T_M^{ij}

$$T_M^{ij} = \sqrt{\frac{s_{ij}^3}{2\mu}} \left\{ \left[\frac{d_{ij}}{s_{ij}} \left(1 - \frac{d_{ij}}{s_{ij}}\right) \right]^{1/2} + \frac{1}{2} \sin^{-1}\left(\frac{2d_{ij}}{s_{ij}} - 1\right) + \frac{\pi}{4} \right\} \qquad (1.3.2)$$

It is easy to see that as $a \to \infty$ the right hand side of (1.2.3) also goes to ∞. However, if we take the limit of the function given by (1.2.2) as $a \to \infty$ it can be shown that the result is a function identical to that appearing in the right hand side of (1.2.8). (This is not immediately obvious and finding this limit involves some analysis. A modified version of L'Hospital's rule can be applied to a suitable representation of the function as a quotient of two functions both tending to zero as $a \to \infty$.) Let us define this limiting function by T_A^{ij}. Hence

$$T_A^{ij} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\,[s_{ij}^{3/2} - (s_{ij} - d_{ij})^{3/2}] \, . \qquad (1.3.3)$$

It can be shown (by examining the derivative of (1.2.2)) that the slope of this function is always negative in the domain $\frac{s_{ij}}{2} < a < \infty$.

If the limit $a \to 0$ is taken for the right hand side of (1.2.6) the result is zero. It can be shown however that on taking the limit of this function as $a \to \infty$ the result is again identical to the function given by (1.2.8) and hence equal to T_A^{ij}. These results are in agreement with what one would expect from the physical situation since $a \to \infty$ represents passing to the limiting case of a parabolic orbit. By taking the derivative of (1.2.6) it can be shown that the slope of this function is positive in the interval $0 < a < \infty$. Figure 2 is a graph of the elliptical functions (1.2.2) and (1.2.3) together with the hyperbolic function (1.2.6) for fixed $\vec{R}_i$ and $\vec{R}_j$ such that $\theta_{ij} < 180^\circ$.

A similar analysis can be made for the case $180^\circ \leq \theta_{ij} < 360^\circ$. In this situation if $a = s_{ij}/2$ is

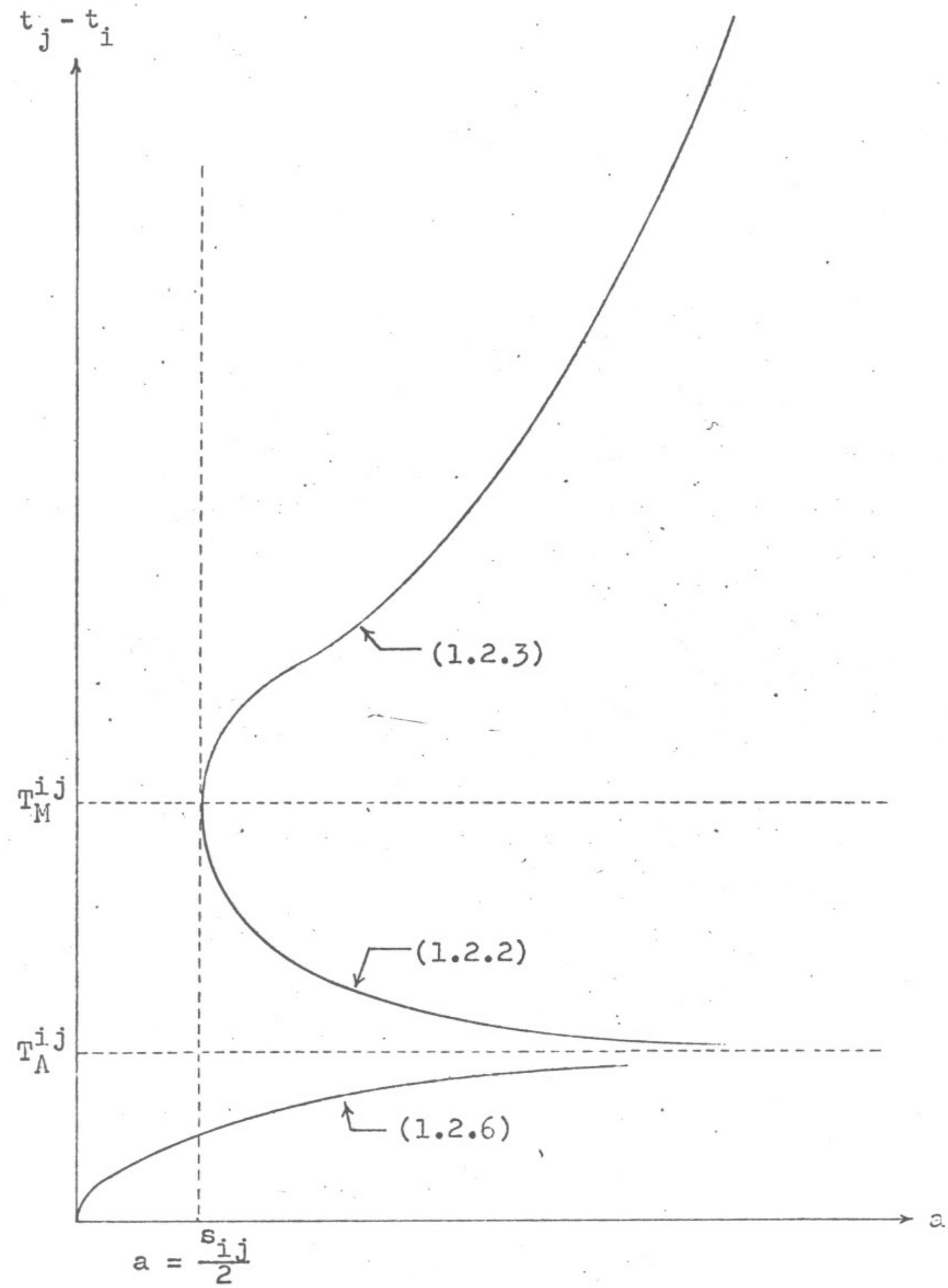


Figure 2. $0 < \theta_{ij} < 180°$

substituted into (1.2.4) and (1.2.5) the results are identical and equal to T_M^{ij} , where in this case

$$T_M^{ij} = \sqrt{\frac{s_{ij}^3}{2\mu}} \left\{ \frac{3\pi}{4} - \left\{ \left[\frac{d_{ij}}{s_{ij}} (1 - \frac{d_{ij}}{s_{ij}}) \right]^{1/2} + \frac{1}{2} \sin^{-1} \left(\frac{2d_{ij}}{s_{ij}} - \right) \right\} \right\} . \quad (1.3.4)$$

If the limits of the functions given by (1.2.5) and (1.2.7) are taken as $a \to \infty$ it can be shown that the result is a function T_A^{ij} identical to that defined by (1.2.9). Hence in this case

$$T_A^{ij} = \frac{1}{3} \sqrt{\frac{2}{\mu}} \left[s_{ij}^{3/2} + (s_{ij} - d_{ij})^{3/2} \right] . \quad (1.3.5)$$

The function (1.2.4) tends to ∞ as $a \to \infty$. The general graphs of the elliptical functions (1.2.4), (1.2.5) and the hyperbolic function (1.2.7) corresponding to the case when $180^\circ \le \theta_{ij} < 360^\circ$ is shown by figure 3.

Consider the class $\mathcal{O}(0,180)$ of all possible conic orbits passing through the fixed vectors $\vec{R}_i$, $\vec{R}_j$ where $0 < \theta_{ij} < 180^\circ$. The domain of the corresponding Lambert and Euler functions will be $a \in [0,\infty]$ and the range of these functions will be $t_j - t_i \in [0,\infty]$. We shall now partition this range into four subregions defined by

$$t_j - t_i \in [0,\infty] = [0,T_A^{ij}) \cup T_A^{ij} \cup (T_A^{ij},T_M^{ij}] \cup (T_M^{ij},\infty] .$$

These subregions will be denoted by

$$\left. \begin{array}{ll} \mathcal{R}_1 = [0,T_A^{ij}) & \mathcal{R}_2 = T_A^{ij} \\ \mathcal{R}_3 = (T_A^{ij},T_M^{ij}] & \mathcal{R}_4 = (T_M^{ij},\infty] \end{array} \right\} \quad (1.3.6)$$

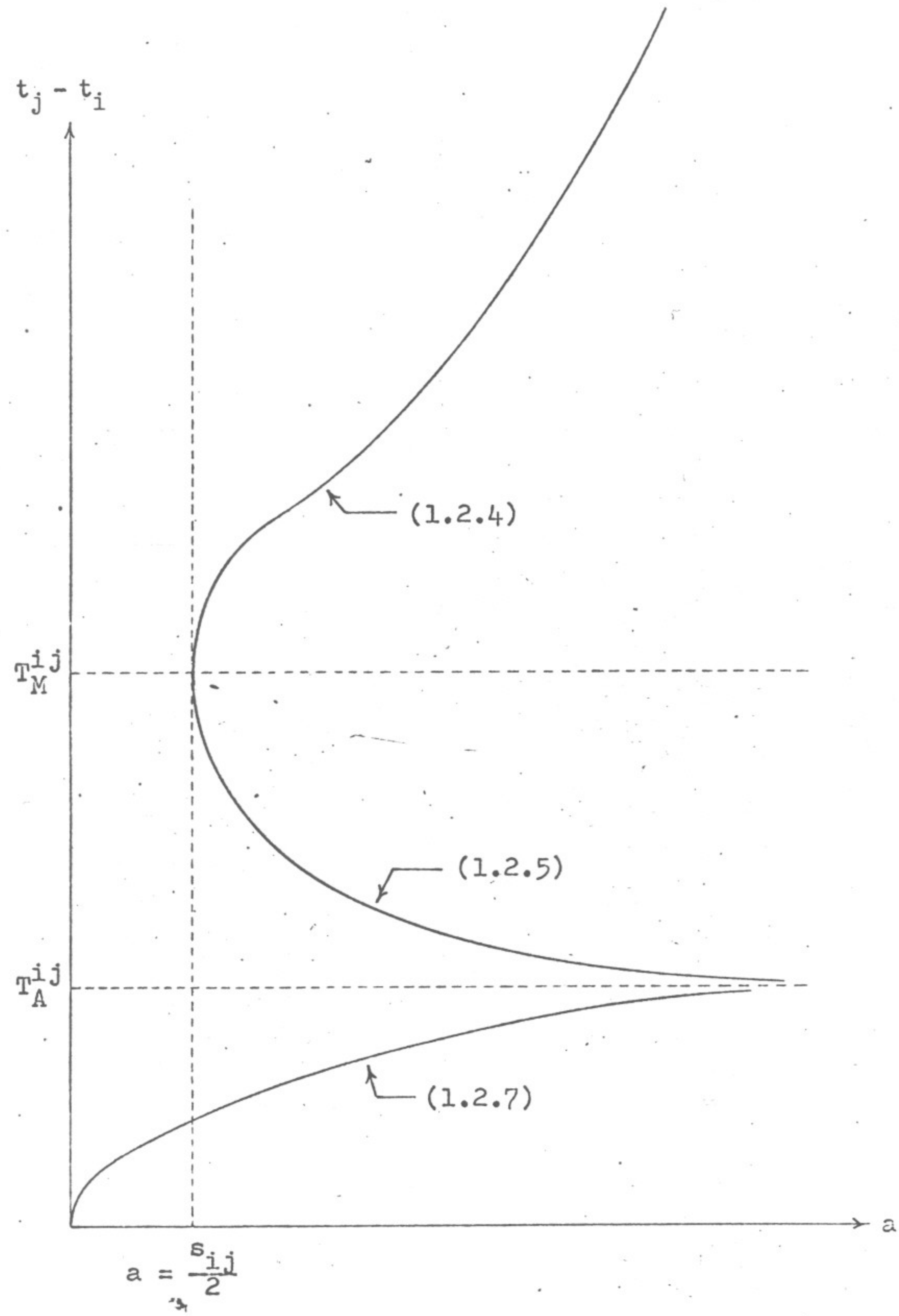


Figure 3. $180° \leq \theta_{ij} < 360°$

Hence, if $t_j - t_i \in \mathcal{R}_1$, the corresponding orbit must be hyperbolic and the correct Lambert function associated with the position vectors $\vec{R}_i$ and $\vec{R}_j$ is uniquely given by (1.2.6). If $t_j - t_i \in \mathcal{R}_2$, only one unique orbit will satisfy this condition. It is parabolic and the corresponding Euler function is given by (1.2.8). If $t_j - t_i \in \mathcal{R}_3$ the orbit must be elliptical and the correct Lambert function is given by (1.2.2). Finally, if $t_j - t_i \in \mathcal{R}_4$, the orbit must also be elliptical with Lambert function given by (1.2.3). A similar partitioning can be carried out for the case $180^\circ \leq \theta_{ij} < 360^\circ$. These partitionings will be key to removing all of the ambiguities in the Lambert-Euler equations.

It is easy to show that the period P corresponding to elliptical orbits is given by

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} . \tag{1.3.7}$$

Consequently, if the integer Z_{ij} denotes the number of complete revolutions the object makes while on an elliptical orbit during the time interval $t_j - t_i$, the term $Z_{ij}P$ must be added to each of the above elliptical functions (1.2.2), (1.2.3), (1.2.4) and (1.2.5).

Let $\mathcal{L}_i^j$ denote the correct Lambert (or Euler) function corresponding to given values of the vectors $\vec{R}_i$ and $\vec{R}_j$. These functions can then be represented uniquely according to the following criterion:

Case 1. $0 < \theta_{ij} \pmod{360^\circ} < 180^\circ$

$$t_j - t_i = \mathcal{L}_i^j = \begin{cases} (1.2.6) \text{ if } & t_j - t_i \in \mathcal{R}_1 \\ (1.2.8) \text{ if } & t_j - t_i \in \mathcal{R}_2 \\ (1.2.2) \text{ if } & (t_j - t_i) - Z_{ij}P \in \mathcal{R}_3 \\ (1.2.3) \text{ if } & (t_j - t_i) - Z_{ij}P \in \mathcal{R}_4 \end{cases} \qquad (1.3.8)$$

Case 2. $180^\circ \leq \theta_{ij} \pmod{360^\circ} < 360^\circ$

$$t_j - t_i = \mathcal{L}_i^j = \begin{cases} (1.2.7) \text{ if } & t_j - t_i \in \mathcal{R}_1 \\ (1.2.9) \text{ if } & t_j - t_i \in \mathcal{R}_2 \\ (1.2.5) \text{ if } & (t_j - t_i) - Z_{ij}P \in \mathcal{R}_3 \\ (1.2.4) \text{ if } & (t_j - t_i) - Z_{ij}P \in \mathcal{R}_4 \end{cases} \qquad (1.3.9)$$

This formulation of Lambert's equations is completely free of ambiguities. Given two vectors $\vec{R}_i$ and $\vec{R}_j$ together with the corresponding flight time $t_j - t_i$ (the direction of motion and Z_{ij} are also assumed known), the correct Lambert function can be identified by the above criterion and used to calculate the orbit's semi-major axis.

1.4 Determination of an Orbit from two Position Vectors and Time of Flight.

Let $\vec{R}_i$ and $\vec{R}_j$ correspond to two given position vectors of an object moving on an elliptical orbit at times t_i and t_j respectively. Suppose $\theta_{ij} = \measuredangle \vec{R}_i, \vec{R}_j < 180^\circ$ and $t_i < t_j$. Since the orbital vector $\vec{e}_{ij}$ lies in the plane of motion defined by $\vec{R}_i$ and $\vec{R}_j$, there exists two scalars q_i and q_j such that

$$\vec{e}_{ij} = q_i\vec{R}_i + q_j\vec{R}_j \qquad (1.4.1)$$

After dot multiplying both sides of this equation by $\vec{R}_i$ and $\vec{R}_j$ respectively, we obtain (with the help of (1.1.21) and (1.1.22)) the system

$$\begin{aligned} q_i R_i^2 + q_j \vec{R}_i \cdot \vec{R}_j &= \ell_{ij} - R_i \\ q_i \vec{R}_i \cdot \vec{R}_j + q_j R_j^2 &= \ell_{ij} - R_j \,, \end{aligned}$$

where ℓ_{ij} denotes the orbit's semi-latus rectum. The solution is

$$\left.\begin{aligned} q_i &= [R_j^2(\ell_{ij}-R_i) - \vec{R}_i\cdot\vec{R}_j(\ell_{ij}-R_j)]/D \\ q_j &= [R_i^2(\ell_{ij}-R_j) - \vec{R}_i\cdot\vec{R}_j(\ell_{ij}-R_i)]/D \end{aligned}\right\} \qquad (1.4.2)$$

where $D = (R_iR_j)^2 - (\vec{R}_i\cdot\vec{R}_j)^2 \neq 0$. Now, it follows from (1.4.1) that

$$e_{ij}^2 = q_i^2R_i^2 + 2q_iq_j\,\vec{R}_i\cdot\vec{R}_j + q_j^2R_j^2 \,.$$

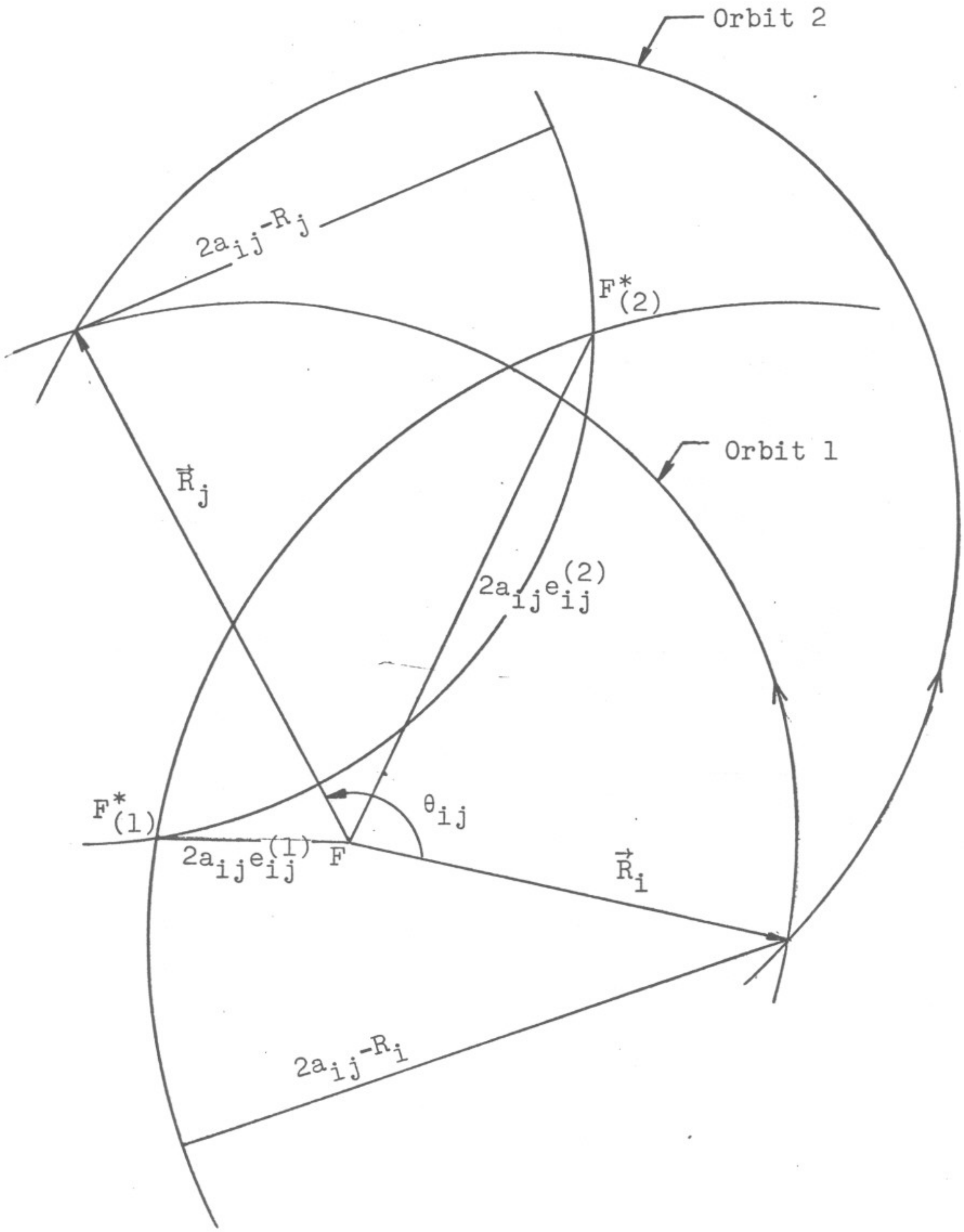
After substituting (1.4.2) into this equation and substituting the result into

$$a_{ij}(1 - e_{ij}^2) = \ell_{ij} , \tag{1.4.3}$$

one can obtain

$$(\ell_{ij}-R_i)^2 R_j^2 - 2(\ell_{ij}-R_i)(\ell_{ij}-R_j)(\vec{R}_i \cdot \vec{R}_j) + (\ell_{ij}-R_j)^2 R_i^2 + \left(\frac{\ell_{ij}}{a_{ij}} - 1\right) D = 0 . \tag{1.4.4}$$

Omitting the algebra, it is easy to show that, for a given value of $a_{ij} > a_m$, this quadratic equation will, in general, admit two real and positive roots, $\ell_{ij}^{(1)}$ and $\ell_{ij}^{(2)}$. We can assume that $\ell_{ij}^{(2)} < \ell_{ij}^{(1)}$. Hence, in view of (1.4.3) these roots will give rise to two possible eccentricities; $e_{ij}^{(1)}$ and $e_{ij}^{(2)}$, which correspond to $\ell_{ij}^{(1)}$ and $\ell_{ij}^{(2)}$ respectively. Consequently, since we assume $\ell_{ij}^{(2)} < \ell_{ij}^{(1)}$, it follows from (1.4.3) that $e_{ij}^{(2)} > e_{ij}^{(1)}$. Hence, there exists two possible orbits passing through $\vec{R}_i$ and $\vec{R}_j$ that have the same value for the semi-major axis. These orbits will correspond to different values for $t_j - t_i$. This has already been observed during our study of the graphs for elliptical orbits shown in figure 2. Omitting the details, it can be shown with the aid of the geometry of figure 4 that the different eccentricities $e_{ij}^{(1)}$, $e_{ij}^{(2)}$ correspond to the Lambert functions (1.2.2) and (1.2.3) respectively. It can be shown by elementary calculations involving the roots of



Fugure 4. Construction of two possible elliptical orbits passing through two fixed points $\vec{R}_i$, $\vec{R}_j$ with constant semi-major axis a_{ij} and $\theta_{ij} < 180^o$

(1.4.4) and making use of (1.4.3) that

$$e_{ij}^{(1)} = \{1 - \frac{2}{d_{ij}^2}(s_{ij}-R_i)(s_{ij}-R_j)[1-x_{ij}y_{ij} + (1-x_{ij}^2)^{1/2}(1-y_{ij}^2)^{1/2}]\}^{1/2} \tag{1.4.5}$$

and

$$e_{ij}^{(2)} = \{1 - \frac{2}{d_{ij}^2}(s_{ij}-R_i)(s_{ij}-R_j)[1-x_{ij}y_{ij}-(1-x_{ij}^2)^{1/2}(1-y_{ij})^{1/2}]\}^{1/2}. \tag{1.4.6}$$

By a similar analysis involving hyperbolic orbits, one can show that the eccentricity functions corresponding to (1.2.6) and (1.2.7) are given by

$$e_{ij}^{(1)} = \{1 + \frac{2}{d_{ij}^2}(s_{ij}-R_i)(s_{ij}-R_j)[x_{ij}y_{ij}+(x_{ij}^2-1)^{1/2}(y_{ij}^2-)^{1/2}-1]\}^{1/2} \tag{1.4.7}$$

and

$$e_{ij}^{(2)} = \{1 + \frac{2}{d_{ij}^2}(s_{ij}-R_i)(s_{ij}-R_j)[x_{ij}y_{ij}-(x_{ij}^2-1)^{1/2}(y_{ij}^2-1)^{1/2}]\}^{1/2} \tag{1.4.8}$$

respectively.

We are now in a position to solve the orbit determination problem of this section. Let $\vec{R}_i$ and $\vec{R}_j$ denote two position vectors of an arbitrary Keplerian path at times t_i and t_j respectively. We shall assume that the direction of motion is known and that $\theta_{ij} < 360^\circ$. Then, by (1.3.8) and (1.3.9), the orbit's semi-major axis can be calculated. The orbit's eccentricity can be determined via (1.4.5)-(1.4.8). The semi-latus rectum ℓ_{ij} can be obtained by (1.1.22). Hence, the orbital vector $\vec{e}_{ij}$ can be obtained by (1.4.1) and (1.4.2). The second orbital vector $\vec{h}_{ij}$ can be obtained by

$$\vec{h}_{ij} = \pm \frac{\vec{R}_i \times \vec{R}_j}{|\vec{R}_i \times \vec{R}_j|} h_{ij} , \qquad (1.4.9$$

where h_{ij} is given by (1.1.26). The positive or negative sign depends upon θ_{ij} and the direction of motion.

Chapter II

Gravity Thrust Space Trajectories

2.1 Notation

It is convenient to represent Gravity Thrust space trajectories by $P_o - P_1 - P_2 - \cdots - P_n$. In this representation P_o denotes the launch planet and P_i $(i = 1,2,\ldots,n-1)$ denotes the i^{th} planet passed. The terminal point is denoted by P_n . Since, for our purposes, no thrusting maneuver is required at P_n , this point need not be a planet. For example, it may be a comet, another free-fall interplanetary space vehicle or a point several astronomical units above the ecliptic plane, which would normally be very difficult to reach via a classical direct transfer trajectory from P_o . Hence, in these representations $n \geq 2$. The following additional notation will be employed throughout this paper:

Σ = heliocentric equational coordinate system (1950.0)

Σ_i = parallel translation of Σ to center of P_i $(i = 1,2,\ldots,n)$

W.R.T. = with respect to

P_i-P_{i+1} = transfer trajectory from P_i to P_{i+1} $(i = 0,1,2,\ldots,n-1)$

t_i = time of closest approach to P_i

$\vec{R}(t)$ = position vector of vehicle W.R.T. Σ at time t

$\vec{R}_i(t)$ = position vector of P_i W.R.T. Σ at time t

τ_i = region of gravitational influence of P_i

$\rho_i(t)$ = position vector of vehicle W.R.T. Σ_i at time t

d_i	=	distance of closest approach to surface of P_i
$\vec{V}_i(t)$	=	velocity vector of P_i W.R.T. Σ at time t
$\vec{V}(t)$	=	velocity vector of vehicle W.R.T. Σ at time t
$\vec{v}_i(t)$	=	velocity vector of vehicle W.R.T. Σ_i at time t
t_{i1}, t_{i2}	=	times at which vehicle enters and leaves τ_i respectively
$\vec{e}_{ij}, \vec{h}_{ij}$	=	orbital vectors of $P_i - P_j$ $(i = 0,1,2,\ldots,n-1;$ $j = i+1)$ W.R.T. Σ respectively
$\vec{e}_i$, $\vec{h}_i$	=	orbital vectors of vehicle inside τ_i W.R.T. Σ_i respectively
a_{ij}, ℓ_{ij}	=	semi-major axis and semi-latus rectum of $P_i - P_j$ $(i = 0,1,2,\ldots,n-1; j = i+1)$ W.R.T. Σ respectively
a_i , ℓ_i	=	semi-major axis and semi-latus rectum of vehicle's orbit inside τ_i W.R.T. Σ_i respectively.

2.2 Fundamental Assumptions

Exact theoretical solutions to the problem of finding a Gravity Thrust trajectory of the form $P_o - P_1 - P_2 - \cdots - P_n$ are not known. This is essentially the unsolved N-body problem. However, our solar system has certain physical characteristics which can be utilized to obtain approximate solutions. All of the planets move about the Sun on nearly constant elliptical paths. Furthermore, when a free-fall vehicle approaches sufficiently close to a passing planet, the vehicle's path (for all practical purposes) becomes hyperbolic relative to the planet. When the vehicle leaves the vicinity of the planet, the Sun begins to dominate its motion, and the path once again becomes elliptical relative to the Sun. The pre-encounter and post-encounter elliptical paths may be entirely different. Indeed, in some cases, the post encounter trajectory will become hyperbolic. Hence, it will be assumed that, at any given instant in time, the vehicle will be moving on a strictly Keplerian path either hyperbolic relative to a nearby planet or elliptical (or hyperbolic) relative to the Sun.

The region of space surrounding a planet in which a vehicle moves on a hyperbolic path relative to it will be called the planet's "gravitational region of influence", or "activity region". It will be taken to be a spherical region of radius ρ^*, given by the formula

$$\rho^* = R_p \left(\frac{m_p}{M_\odot}\right)^{2/5} . \qquad (2.2.1)$$

In this formula m_p and $M_\odot$ denote the masses of the planet and Sun respectively. The distance between the Sun and the planet is denoted by R_p. The activity sphere can be shown (8) to be the locus of points where (viewed from Σ_i) the perturbation on the vehicle from the Sun is approximately equal to the perturbation from the planet (viewed from Σ). Table A is a list of approximate values for ρ^*, corresponding to the various planets as calculated by setting R_p equal to the planet's semi-major axis.

Table A. Approximate Radii of Planetary Activity Spheres

Planet	R_p(A.U.)	ρ^*(A.U.)
Mercury	0.387	0.000747
Venus	0.723	0.00412
Earth	1.000	0.00618
Mars	1.524	0.00387
Jupiter	5.203	0.387
Saturn	9.539	0.349
Uranus	19.182	0.347
Neptune	30.058	0.577
Pluto	39.601	0.240

Figure 5 is a simplified two-dimensional picture of the motion of a vehicle on a Gravity Thrust trajectory during the time it is in the vicinity of a particular planet p_i. The vehicle enters and leaves p_i's activity sphere at points A and C at times t_{i1} and t_{i2} respectively. When it enters τ_i, the planet is at D ; and when it leaves τ_i, the planet is at F . The points B and E correspond

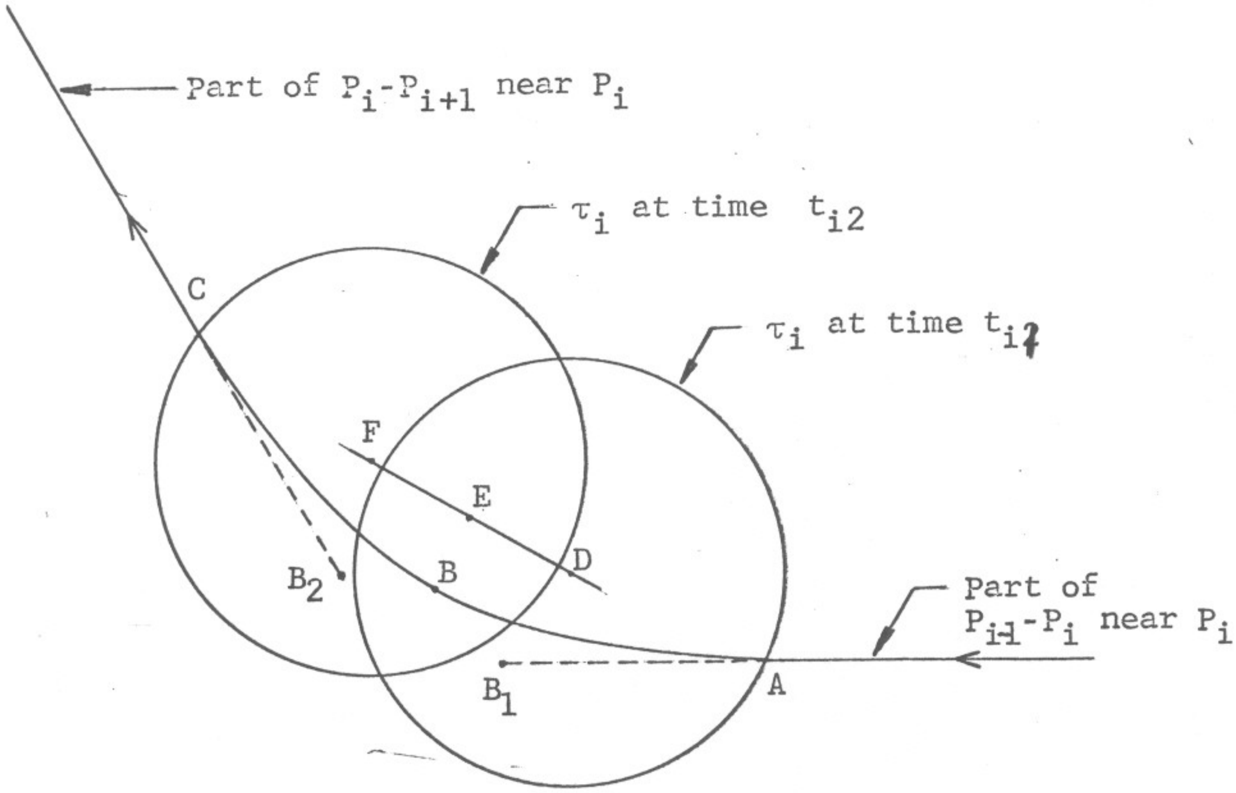


Figure 5. Vehicle's trajectory near P_i with respect to the Sun

respectively to the positions of the vehicle and the planet at the time of closest approach t_i . Suppose the perturbation of P_i is neglected. Then a vehicle moving on an interplanetary orbit corresponding to $P_{i-1} - P_i$ would be at B_1 at time t_i . Similarly, a vehicle moving on an orbit corresponding to $P_i - P_{i+1}$ would be at B_2 and at time t_i. Since figure 5 is drawn with respect to Σ , the trajectory between A and C is not Keplerian. Figure 6 describes this part of the trajectory with respect to Σ_i . In this frame of reference, the perturbing planet is at rest, and the trajectory of the vehicle is hyperbolic.

In addition to our principle assumption of Keplerian motion, we shall add two additional assumptions which are relatively minor. Since we are not concerned with the launch and terminal phases of our trajectories, the perturbation of P_o and the possible perturbation of P_n at the initial and terminal points will be neglected. Furthermore, we shall take the center of P_o and P_n as the initial and terminal points respectively.

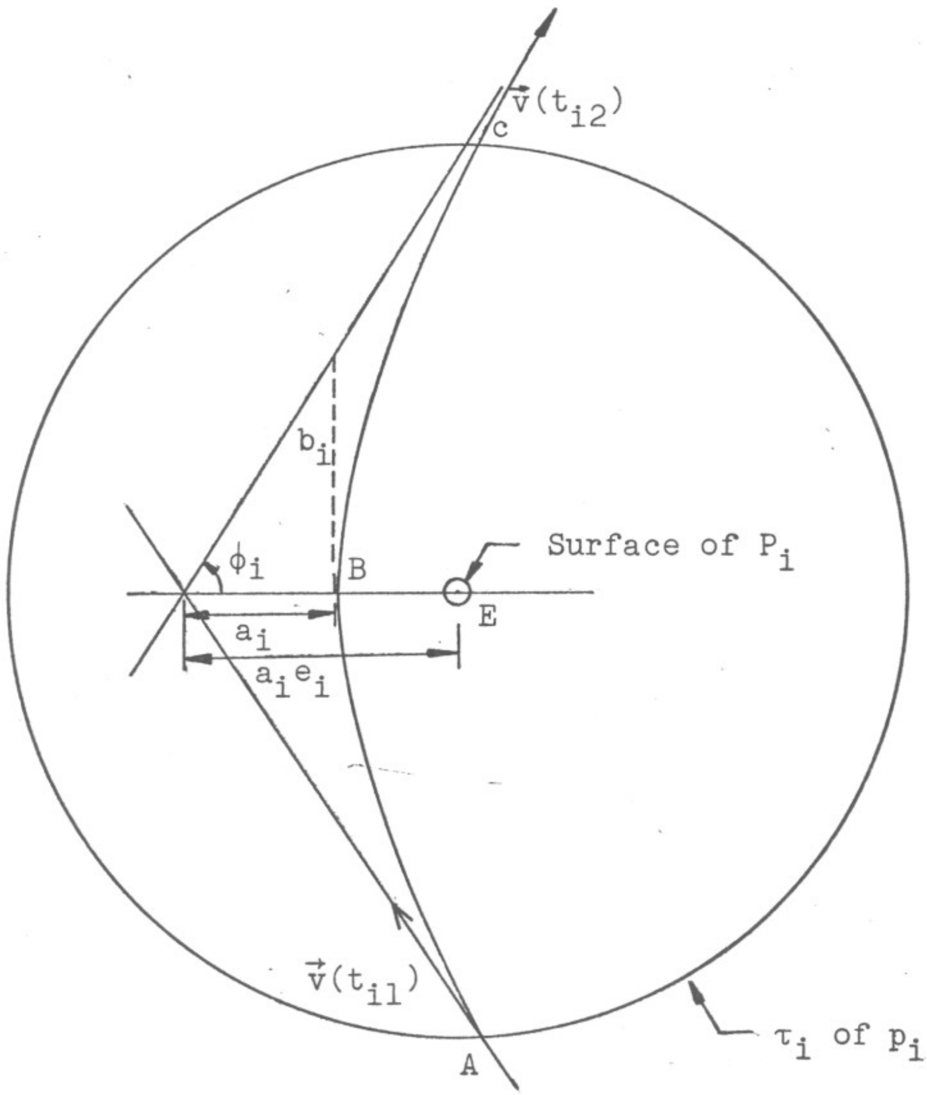


Figure 6. Hyperbolic Encounter Trajectory

2.3 The Fundamental Gravity Thrust Trajectory Problem (Boundary Conditions)

Let P_o - P_1 - P_2 ····· P_n represent a given Gravity Thrust trajectory profile. By this terminology we shall mean that the planets P_i $(i = 0,1,2,\ldots,n-1)$ are specified, as well as the order in which they are to be encountered. If P_n is not a planet, its ephemeris or coordinates are assumed to be known. By physically realizable trajectories we mean those trajectories which have positive distances of closest approach d_i at each encountered planet P_i $(i = 1,2,\ldots,n-1)$. The problem of determining a physically realizable trajectory of a given profile and given launch date t_o and first planetary closest approach date t, will be called the "fundamental Gravity Thrust trajectory problem of astrodynamics." Although this formulation of the problem does not uniquely determine these trajectories, it alleviates what might otherwise be rather difficult analytical problems.

One example of a Gravity Thrust trajectory profile is Earth-Venus-Earth. These are simple round trip free-fall reconnaissance trajectories of Venus. More ambitious profiles can be obtained by utilizing the more powerful pertubation fields of the outer planets. An example would be Earth-Venus-Mars-Earth-Saturn-Pluto-Jupiter-Earth (page 39, Ref. 6). In general, the total number of different Gravity Thrust trajectory profiles of the form P_o-P_1-P_2-····-P_n having n planetary encounters is 9^{n+1}.

This concludes the conceptual formulation of Gravity Thrust trajectories. The analytical study can now be undertaken.

2.4 The Energy Exchange Equation

Since our pre-encounter and post-encounter trajectories will be different Keplerian paths relative to the Sun, the energy required to realize these trajectory changes must come from a gravitational interaction with the passing planet. The amount of energy which will be exchanged depends upon the vehicle's approach trajectory.

Suppose $t_{i1} < t < t_{i2}$. Then, according to the notation of section 2.1, it follows that

$$\vec{R}(t) = \vec{R}_i(t) + \vec{\rho}_i(t) .$$

Hence,

$$\vec{V}(t) = \vec{V}_i(t) + \vec{v}_i(t) \qquad (2.4.1)$$

Upon squaring both sides of this equation we obtain

$$V^2(t_{ij}) = V_i^2(t_{ij}) + 2\vec{V}_i(t_{ij})\cdot\vec{v}_i(t_{ij}) + v_i^2(t_{ij}) \cdot (j=1,2) \qquad (2.4.2)$$

However, by making use of the energy equation for hyperbolic orbits (1.1.23), we may write

$$v_i^2(t_{ij}) = \mu_i\left(\frac{2}{\rho_i(t_{ij})} + \frac{1}{a_i}\right) .$$

But since $\rho_i(t_{ij}) = \rho_i^*$ $(j = 1,2)$, we conclude that

$$v_i^2(t_{i1}) = v_i^2(t_{i2}) . \qquad (2.4.3)$$

Substituting this result into the difference of the two equations given by (2.4.2), we obtain

$$V^2(t_{i2}) - V^2(t_{i1}) = 2\vec{V}_i\cdot[\vec{v}_i(t_{i2}) - \vec{v}_i(t_{i1})] . \qquad (2.4.4)$$

In obtaining this equation, we have assumed $\vec{V}_i(t_{i1}) = \vec{V}_i(t_{i2}) = \vec{V}_i$. Since the arc swept out by the planet about the Sun during the time the vehicle spends in its sphere of influence, is relatively short, this assumption will introduce very little error. It is not, however, essential in our analysis. Now, in view of (2.4.1), we may write

$$\vec{v}_i(t_{i2}) - \vec{v}_i(t_{i1}) = \vec{V}(t_{i2}) - \vec{V}(t_{i1}) \ .$$

Upon substituting this result into (2.4.4), we obtain

$$\frac{1}{2}[V^2(t_{i2}) - V^2(t_{i1})] = \vec{V}_i \cdot [\vec{V}(t_{i2}) - \vec{V}(t_{i1})] \ . \qquad (2.4.5)$$

This equation represents the amount of energy exchanged between the planet P_i and the vehicle (of unit mass). It will be referred to as the Energy Exchange equation. It will play a major role in the solution of all our trajectory problems.

2.5 A Solution to the Fundamental Gravity Thrust Trajectory Problem

Let S denote the center of the Sun. Then, on an interplanetray scale, the vectors $\vec{SE}$, $\vec{SB}_1$, and $\vec{SB}_2$ (see figure 4) will be almost identical. With this assumption, the initial interplanetary transfer trajectory P_{i-1}—P_i (for the time being, the subscript i will be equal to 1) can be obtained by the method given in section 1.4. The position vectors $\vec{R}_{i-1}$ and $\vec{R}_i$ of P_{i-1} and P_i at times t_{i-1} and t_i respectively can be obtained from a planetary ephemeris. The resulting orbital vectors $\vec{e}_{i-1,i}$ and $\vec{h}_{i-1,i}$ will be very close to the true orbital vectors. Let t_{i+1} represent a trial value for this date. Then, by making use of an ephemeris for P_{i+1} , the corresponding trial vector $\vec{R}_{i+1}$ can be calculated. Hence, the corresponding trial orbital vectors $\vec{e}_{i,i+1}$, $\vec{h}_{i,i+1}$ can also be obtained via section 1.4. Now, in view of the orbit energy equation (1.1.23), it follows that

$$V^2(t_{i2})-V^2(t_{i1}) = \mu_s\left(\frac{1}{a_{i-1,i}} - \frac{1}{a_{i,i+1}}\right), \qquad (2.5.1)$$

where $\mu_s = GM_\odot$. Now, by making use of (1.1.18), we may write

$$\vec{V}_i\cdot[\vec{V}(t_{i2})-\vec{V}(t_{i1})] = \vec{V}_i\cdot[\vec{h}_{i,i+1}\times(\vec{e}_{i,i+1}+\hat{R}_i)-\vec{h}_{i-1,i}\times(\vec{e}_{i-1,i}+\hat{R}_i)] \;. \qquad (2.5.2)$$

It is clear that the right hand sides of (2.5.1) and (2.5.2) are functions of t_{i+1} . Let $G(t_{i+1})$ denote the scalar function which is defined by

$$G(t_{i+1}) = \mu_s\left[\frac{1}{a_{i-1,i}} - \frac{1}{a_{i,i+1}}\right] - 2\vec{V}_i \cdot [\vec{h}_{i,i+1} \times (\vec{e}_{i,i+1} + \hat{R}_i)$$

$$- h_{i-1,i} \times (\vec{e}_{i-1,i} + \hat{R}_i)] \; . \qquad (2.5.3)$$

Consequently, in view of the energy exchange equation (2.4.5), a solution for t_{i+1} can be obtained by solving the equation

$$G(t_{i+1}) = 0 \; . \qquad (2.5.4)$$

It is obvious that solutions to (2.5.4) can not be easily obtained by purely analytical processes. The use of a high speed digital computer will be absolutely essential. Assuming a computer is available, a solution can be obtained as follows: Let t_{i+1} be an initial value not too far removed in time from t_i , so that it can be assumed that (2.5.4) will not be satisfied for any t in the interval $t_i < t < t_{i+1}$. Let Δt_{i+1} denote a sufficiently small time interval such that (2.5.4) is unlikely to yield two distinct solutions within the time interval Δt_{i+1} . Then t_{i+1} is incremented by steps of $(10^{-0})\Delta t_{i+1}$. When the sign change occurs, $(10^{-0})\Delta t_{i+1}$ is subtracted from t_{i+1} , and the incrementing continues with the still smaller step size $(10^{-1})\Delta t_{i+1}$. This process can be repeated until a value of t_{i+1} is found which satisfies $|G(t_{i+1})| < \varepsilon$ for any arbitrarily small number ε . (Of course ε will have to be compatible with the number of significant decimal places carried along in the computations.) It should be noted that finding a solution to (2.5.4) is equivalent to finding a transfer trajectory $P_i - P_{i+1}$ which will yield an asymptotic departure speed from P_i equal to the asymptotic approach speed (i.e., such that $v(t_{i2}) = v(t_{i1})$.

See section 2.4).

Now, since

$$\vec{V}(t_{i1}) = \vec{h}_{i-1,i} \times (\vec{e}_{i-1,i} + \hat{R}_i)$$

and

$$\vec{V}(t_{i2}) = \vec{h}_{i,i+1} \times (\vec{e}_{i,i+1} + \hat{R}_i) ,$$

the asymptotic approach and departure velocities relative to P_i can be obtained by

$$\vec{v}(t_{i1}) = \vec{V}(t_{i1}) - \vec{V}_i$$

and

$$\vec{v}(t_{i2}) = \vec{V}(t_{i2}) - \vec{V}_i$$

respectively. Therefore, by making use of the energy equation (1.1.23), the semi-major axis a_i of the encounter trajectory can be obtained by

$$a_i = \frac{\mu_i \rho_i^*}{v^2(t_{i1}) - 2\mu_i} . \tag{2.5.5}$$

Moreover, by studying figure 5, we find

$$\tan \emptyset_i = \frac{b_i}{a_i} .$$

Consequently, since

$$e_i = \sqrt{1 + \left(\frac{b_i}{a_i}\right)^2} ,$$

it follows that

$$\cos \emptyset_i = \frac{1}{e_i} . \tag{2.5.6}$$

From the figure, we may also write

$$\vec{v}(t_{i1}) \cdot \vec{v}(t_{i2}) = v(t_{i1}) v(t_{i2}) \cos 2(\tfrac{\pi}{2} - \emptyset_i) .$$

Combining this result with (2.5.6), we obtain

$$e_i = \left\{ \frac{2v^2(t_{i1})}{v^2(t_{i1}) - \vec{v}(t_{i1})\cdot\vec{v}(t_{i2})} \right\}^{1/2} . \qquad (2.5.7)$$

The distance of closest approach to the surface of P_i can now be easily computed by

$$d_i = a_i(e_i - 1) - r_i , \qquad (2.5.8)$$

where r_i denotes the radius of P_i . If d_i turns out to have a negative value, the solution t_{i+1} which was found for (2.5.4) is discarded. The incrementing is continued from a new initial value $t_{i+1} + \Delta t_{i+1}$, where t_{i+1} was the previous solution. When a new solution to (2.5.4) is obtained, the above calculations are repeated so that a new distance of closest approach is obtained. This process is repeated until a positive value for d_i is obtained, or until $t_{i+1} - t_i$ becomes unreasonably long.

Suppose a positive value for d_i is obtained. In this case the orbital vector $\vec{e}_i$ can be immediately calculated from

$$\vec{e}_i = \frac{\vec{v}(t_{i1}) - \vec{v}(t_{i2})}{|\vec{v}(t_{i1}) - \vec{v}(t_{i2})|} \; e_i , \qquad (2.5.9)$$

which follows directly from figure 5. Also, since h_i can be obtained by

$$h_i = \sqrt{\frac{\mu_i}{a_i(e_i^2 - 1)}} , \qquad (2.5.10)$$

the second orbital vector can be obtained by

$$\vec{h}_i = \frac{\vec{v}(t_{i1}) \times \vec{v}(t_{i2})}{|\vec{v}(t_{i1}) \times \vec{v}(t_{i2})|} \; h_i \; . \tag{2.5.11}$$

During this entire process, the values of t_{i1} and t_{i2} were completely unknown. These times can now be calculated by making use of the energy equation (1.1.23). Hence,

$$t_{i2} - t_i = \frac{1}{\sqrt{\mu_i}} \int_{q_i}^{\rho_i^*} \frac{dr}{\sqrt{\frac{2}{R} + \frac{1}{a}}} \quad ,$$

where $q_i = a_i(e_i - 1)$. After performing the integration we find

$$t_{i2} - t_i = \sqrt{\frac{a_i}{\mu_i}} \, (\alpha_i - \beta_i) \; , \tag{2.5.12}$$

where

$$\alpha_i = \sqrt{\rho_i^{*2} + 2a_i\rho_i^* - a_i^2(e_i^2 - 1)}$$

$$\beta_i = a_i \; \log[\frac{1}{a_i e_i} \, (\alpha_i + \rho_i^* + a_i)] \; .$$

Since $t_{i2} - t_i = t_i - t_{i1}$, we can obtain t_{i1} by

$$t_{i1} = t_i - (t_{i2} - t_i) \; . \tag{2.5.13}$$

By referring to figure 5 once again we notice that, if $t_i < t < t_{i2}$, the position vector $\vec{\rho}_i(t)$ can be calculated by

$$\vec{\rho}_i(t) = \rho_i(t)[\cos \theta_i \hat{e}_i + \sin \theta_i (\hat{h}_i \times \hat{e}_i)] \; ,$$

where $\theta_i = \sphericalangle \, \hat{e}_i \, , \vec{\rho}_i(t)$. Hence, the vectors $\vec{\rho}_i(t_{i1})$ and $\vec{\rho}_i(t_{i2})$ can be calculated by

$$\vec{\rho}_i(t_{i1}) = \rho_i^*[\cos\theta_i^*\hat{e}_i - \sin\theta_i^*(\hat{h}_i \times \hat{e}_i)] \tag{2.5.14}$$

and

$$\vec{\rho}_i(t_{i2}) = \rho_i^*[\cos\theta_i^*\hat{e}_i + \sin\theta_i^*(\hat{h}_i \times \hat{e}_i)] \ , \tag{2.5.15}$$

where

$$\cos\theta_i^* = \left[\frac{\ell_i}{\rho_i^*} - 1\right]\frac{1}{e_i}$$

$$\sin\theta_i^* = \sqrt{1 - \cos^2\theta_i^*} \ .$$

In the above analysis $i = 1$. Hence, these calculations only represent a possible solution for the first part of the trajectory $P_o - P_1 - P_2$. However, by successively incrememting i to $i+1$ and repeating the above calculations beginning with finding an acceptible solution to (2.5.4) and continuing through (2.5.15) until $i = n-1$) is reached, a complete physically realizable trajectory $P_o - P_1 - P_2 - \cdots - P_n$ will be obtained. (A detailed step by step set of computer instructions corresponding to this solution can be found on page 41 ff of reference 6).

In determining the above solution, a certain amount of error was introduced at each step by assuming $\vec{SE} = \vec{SB}_1 = \vec{SB}_2$. This was necessary because the precise points at which the vehicle enters and leaves each successive activity sphere relative to Σ can not be calculated until $\vec{\rho}_i(t_{i1})$ and $\vec{\rho}_i(t_{i2})$ $(i = 1,2,\ldots,n-1)$ are known. However, this error will in fact be extremely small so that, for all practical purposes, the above solution can be assumed to be exact.

Extensive computations at J.P.L. have revealed that the solution gives remarkably accurate trajectories. For a more detailed discussion concerning accuracy, see pages 20 and 52 of reference 9. However, it is possible to calculate a solution to any desired degree of accuracy by using only the assumptions of section 2.2. This can be achieved by successive approximations.

Let the above solution correspond to the first approxomation. In order to obtain the k+l'th approximation, the transfer orbits $P_{i-1} - P_i$ are obtained by substituting the vectors $\vec{R}(t_{i-1,2}) = \vec{R}_{i-1}(t_{i-1,2}) + \vec{\rho}_{i-1}(t_{i-1,2})$ and $\vec{R}(t_{i,1}) = \vec{R}_i(t_{i,1}) + \vec{\rho}_i(t_{i,1})$, together with the flight times $t_{i,1} - t_{i-1,2}$, into the method of section 1.4, where the vectors on the right hand sides are obtained by the k'th approximation. All of the above calculations (2.5.4) - (2.5.15) must be repeated $(i = 1,2,\ldots n-1)$ for each successive approximation. It is difficult to construct an analytical proof that this process will converge to a solution. On purely physical grounds, however, it is clear that the approximations will converge and converge very rapidly. In practice, however, this is unnecessary since the first approximation is sufficiently accurate to enable the exact orbit to be determined which corresponds to the actual physical case where all the perturbations of all the planets in the solar system act on the vehicle simultaneously (page 52, reference 9).

Chapter III

Some Extremal Problems in Gravity Thrust Trajectory Design

3.1 Determining the Planetary Approach Trajectory which will Extremize the Post-Encounter Energy

In this chapter we shall be concerned with the problem of determining the various planetary approach trajectories corresponding to a given initial transfer $P_o - P_1$, which will extremize certain physical properties of the post encounter trajectory relative to the Sun. Hence, unless otherwise stated, the initial transfer, determined by $\vec{R}_o(t_o)$ and $\vec{R}_1(t_1)$ from the initial data (i.e., P_o, P_1, t_o, t_1) as discussed in section 1.4), is assumed to have been accomplished. The following notation will be used throughout this chapter:

$\vec{R}_p = (R_1, R_2, R_3)$ = position vector of P_1 at time t_1 W.R.T. Σ

$\vec{V}_p = (u_1, u_2, u_3)$ = velocity vector of P_1 at time t_1 W.R.T. Σ

$\vec{V}_1 = (V_1, V_2, V_3)$ = asymptotic approach velocity vector of vehicle at time t_{11} W.R.T. Σ

$\vec{V}_2 = (x, y, z)$ = asymptotic approach velocity vector of vehicle at time t_{12} W.R.T. Σ

$\vec{v}_1 = (v_1, v_2, v_3)$ = asymptotic approach velocity vector of vehicle at time t_{11} W.R.T. Σ_1

$\vec{v}_2$ = asymptotic departing velocity vector of vehicle at time t_{12} W.R.T. Σ_1

d = distance of closest approach to surface of P_1

r_p = radius of P_1

Let $F_1(x,y,z)$ denote a scalar function of $\vec{V}_2(t_{12})$ defined by

$$F_1(x,y,z) = V_2^2 - V_1^2 - 2\vec{V}_p \cdot (\vec{V}_2 - \vec{V}_1) \ .$$

Hence, in view of the above notation,

$$F_1(x,y,z) = x^2 + y^2 + z^2 - 2(u_1 x + u_2 y + u_3 z) + \sigma \ , \qquad (3.1.1)$$

where σ is a constant defined by

$$\sigma = 2\,\vec{V}_p \cdot \vec{V}_1 - V_1^2 \ . \qquad (3.1.2)$$

This constant depends upon the initial transfer trajectory $P_o - P_1$, which we assume to be already determined. Now, in view of the energy exchange equation (2.4.5), the class of all possible post encounter trajectories can be characterized by

$$F_1(x,y,z) = 0 \ . \qquad (3.1.3)$$

The total energy E of a post encounter trajectory is given by

$$E = \frac{1}{2} V_2^2 - \frac{\mu_s}{R_p} \ . \qquad (3.1.4)$$

We shall now determine those velocity vectors (x,y,z) which will extremize E subject to the constraining condition $F_1(x,y,z) = 0$. This can be done most conveniently by the Lagrange multiplier method. Hence, the extremals of E will be solutions to the following system of equations:

$$\frac{\partial E}{\partial x} - \lambda_1 \frac{\partial F_1}{\partial x} = 0$$

$$\frac{\partial E}{\partial y} - \lambda_1 \frac{\partial F_1}{\partial y} = 0$$

$$\frac{\partial E}{\partial z} - \lambda_1 \frac{\partial F_1}{\partial z} = 0$$

$$F_1 = 0$$

The first three equations yield

$$\left.\begin{aligned} x &= \lambda_o\, u_1 \\ y &= \lambda_o\, u_2 \\ z &= \lambda_o\, u_3 \end{aligned}\right\} \qquad (3.1.5)$$

where

$$\lambda_o = \frac{2\lambda_1}{2\lambda_1 - 1} \quad .$$

Upon substituting these equations into the fourth equation, we obtain a quadratic equation in λ_o , with solutions given by

$$\lambda_o = 1 \pm \sqrt{1 - \frac{\sigma}{V_p^2}} \quad .$$

By making use of the fact that $\vec{v}_2 = \vec{V}_2 - \vec{V}_p$ together with (3.1.2), we obtain

$$\lambda_o = 1 \pm \frac{v}{V_p} \quad ,$$

where $|\vec{v}_2| = |\vec{v}_1| = v$. Substituting these values of λ_o into (3.1.5), we obtain two vectors which extremize E . These are

$$\vec{V}_2 = (1 + \frac{v}{V_p})\vec{V}_p \tag{3.1.6}$$

and

$$\vec{V}_2 = (1 - \frac{v}{V_p})\vec{V}_p . \tag{3.1.7}$$

The magnitude of these velocity vectors are

$$V_2 = V_p + v$$

and

$$V_2 = |V_p - v|$$

respectively. Consequently, the velocity given by (3.1.6) will maximize the energy of the post-encounter trajectory. On the other hand, the second velocity vector given by (3.1.7) will minimize the post-encounter trajectory's energy. It is interesting to notice that if $v = V_p$, then $\vec{V}_2$ given by (3.1.7) will be zero, and the vehicle will fall directly into the Sun. Also, if $V_p < v$, a departing velocity given by (3.1.7) will enable the vehicle to reach a retrograde orbit about the Sun (assuming $d > 0$).

Let $\Phi = \measuredangle \vec{V}_p, \vec{v}_1$. Then, for a maximum energy post-encounter trajectory, the velocity increment $\vec{\Delta V} = \vec{V}_2 - \vec{V}_1$, supplied by the perturbing planet, can be expressed as

$$\Delta V = [(\vec{V}_2-\vec{V}_1)^2]^{1/2} = v\sqrt{2(1-\cos\Phi)} .$$

Hence, the maximum possible velocity which can be given to the vehicle is

$$\Delta V_{max} = 2v . \tag{3.1.8}$$

After selecting which post encounter velocity vector is desired (e.g., (3.1.6) or (3.1.7)), the required planetary approach trajectory can be easily computed. For example, according to section 2.5, we obtain

$$a_1 = \frac{\mu_1 \rho_1^*}{v^2 - 2\mu_1} \quad , \tag{3.1.9}$$

where $v = \sqrt{(\vec{v}_1 - \vec{v}_1)^2} = \sqrt{(\vec{V}_1 - \vec{V}_p)^2}$.

$$e_1 = \left\{ \frac{2v^2}{v^2 - \vec{v}_1 \cdot \vec{v}_2} \right\}^{1/2} \tag{3.1.10}$$

where $\vec{v}_1 \cdot \vec{v}_2 = (\vec{V}_1 - \vec{V}_p) \cdot (\vec{V}_2 - \vec{V}_p)$. The distance of closest approach d_1 , given by

$$d = a_1(e_1 - 1) - r_p \tag{3.1.11}$$

may be negative. For large planets such as Jupiter, Saturn, Nepture, and Uranus, however, this distance will be usually positive. In this case,

$$\vec{e}_1 = \frac{\vec{V}_1 - \vec{V}_2}{|\vec{V}_1 - \vec{V}_2|} e_1 \tag{3.1.12}$$

and

$$\vec{h}_1 = \frac{(\vec{V}_1 - \vec{V}_p) \times (\vec{V}_2 - \vec{V}_p)}{|(\vec{V}_1 - \vec{V}_p) \times (\vec{V}_2 - \vec{V}_p|} h_1 \quad , \tag{3.1.13}$$

where

$$h_1 = \sqrt{\frac{\mu_1}{a_1(e_1^2 - 1)}} \quad . \tag{3.1.14}$$

For relatively low mass planets such as Venus and Mars, the maximum energy post-encounter trajectory corresponding to (3.1.6) will generally result in negative distances of closest approach. Hence, we shall now consider the problem of extremizing the post-encounter trajectory's energy when the distance of closest approach is specified. This will introduce another constraining condition which $\vec{V}_2$ must satisfy.

Let d be a given distance of closest approach. Then, in view of (3.1.11), we have

$$e_1 = 1 + \frac{d + r_p}{a_1} . \qquad (3.1.15)$$

However, since $\vec{V}_1$ is known, the asymptotic approach speed v can be obtained by

$$v = |\vec{V}_1 - \vec{V}_p| .$$

Hence, the semi-major axis a_1 of the hyperbolic encounter trajectory can be determined by (3.1.9). Therefore, in view of (3.1.15), specifying the distance of closest approach is equivalent to specifying the eccentricity e_1 . By making use of (3.1.10), we obtain

$$\vec{v}_1 \cdot \vec{v}_2 = v^2\left(1 - \frac{2}{e_1^2}\right) .$$

This equation can be expressed as

$$F_2(x,y,z) = 0 , \qquad (3.1.16)$$

where

$$F_2(x,y,z) = v_1 x + v_2 y + v_3 z - \gamma \qquad (3.1.17)$$

and

$$\gamma = \vec{v}_1 \cdot \vec{V}_p + v^2\left(1 - \frac{2}{e_1^2}\right) .$$

To extremize E with two constraining equations by Lagrange's multiplier method requires the use of two undetermined multipliers λ_1 and λ_2. The system of equations which will determine the extremal vectors will now be

$$\frac{\partial E}{\partial x} - \lambda_1 \frac{\partial F_1}{\partial x} - \lambda_2 \frac{\partial F_2}{\partial x} = 0$$

$$\frac{\partial E}{\partial y} - \lambda_1 \frac{\partial F_1}{\partial y} - \lambda_2 \frac{\partial F_2}{\partial y} = 0$$

$$\frac{\partial E}{\partial z} - \lambda_1 \frac{\partial F_1}{\partial z} - \lambda_2 \frac{\partial F_2}{\partial z} = 0$$

$$F_1 = 0$$

$$F_2 = 0 .$$

The first three equations reduce to

$$\left.\begin{aligned} x &= \eta_1 v_1 - \eta_2 u_1 \\ y &= \eta_1 v_2 - \eta_2 u_2 \\ z &= \eta_1 v_3 - \eta_2 u_3 \end{aligned}\right\} , \qquad (3.1.18)$$

where

$$\eta_1 = \frac{\lambda_2}{1-2\lambda_1} \qquad \eta_2 = \frac{2\lambda_1}{1-2\lambda_1} .$$

Substituting these equations into the second constraining equation, one finds

$$\eta_1 = \frac{\gamma + (\vec{V}_p \cdot \vec{v}_1)\eta_2}{v^2} .$$

When these results are substituted into the first constraining equation, we obtain the solutions

$$\eta_1 = \frac{v^2}{e_1^2}\left[e_1^2 - 2 + (e_1^2 - 1)^{1/2} \tan\psi \right]$$

$$\eta_2 = \frac{2v(e_1^2 - 1)^{1/2}}{V_p\, e_1^2 \sin\psi} - 1$$

and

$$\eta_1 = \frac{v^2}{e_1^2}\left[e_1^2 - 2 - (e_1^2 - 1)^{1/2} \tan\psi \right]$$

$$\eta_2 = \frac{-2v(e_1^2 - 1)^{1/2}}{V_p\, e_1^2 \sin\psi} - 1 \quad ,$$

where $\psi = \measuredangle\, \vec{V}_p, \vec{V}_1$. The desired extremal vectors can now be obtained by substituting these values of η_1 and η_2 into (3.1.18). The required approach trajectories can be determined via (3.1.12), (3.1.13), and (3.1.14).

3.2 Determining Planetary Approach Trajectories which Extremize Post-Encounter Perihelion and Aphelion Distance

In the previous section we have seen that a free-fall space vehicle approaching a sufficiently strong planetary gravitational field could use the field to achieve a retrograde post-encounter trajectory about the Sun. In order to visualize the required $\vec{V}_2$ to achieve this orbit and its relation to all other possible post-encounter trajectories corresponding to a given P_o - P_1 transfer, it is convenient to view $F_1(x,y,z) = 0$ as a surface in a Cartesian velocity space Σ_v . This surface is a sphere of radius v centered at $\vec{V}_p$. Figure 7 illustrates this surface for a particular P_o - P_1 transfer trajectory, where $v > V_p$. In the figure, $\vec{v}_2^{(1)}$ and $\vec{v}_2^{(2)}$ represent the two extremal velocity vectors (with respect to P_1) which correspond to (3.1.6) and (3.1.7) respectively. The retrograde orbit corresponds to $v_2^{(2)}$. An arbitrary post-encounter velocity vector is shown by $\vec{V}_2$. Consequently, when $v > V_p$, it is possible to send the vehicle either onto a retrograde orbit or onto an impact course with the Sun. Either of these two maneuvers would be extremely difficult to achieve by an ordinary on-board rocket engine.

A solar impact or deep space trajectory would be useful for an unmanned instrumented probe designed to measure environmental properties of space in regions very close to or very far from the Sun. If $v > V_p$, it will be possible

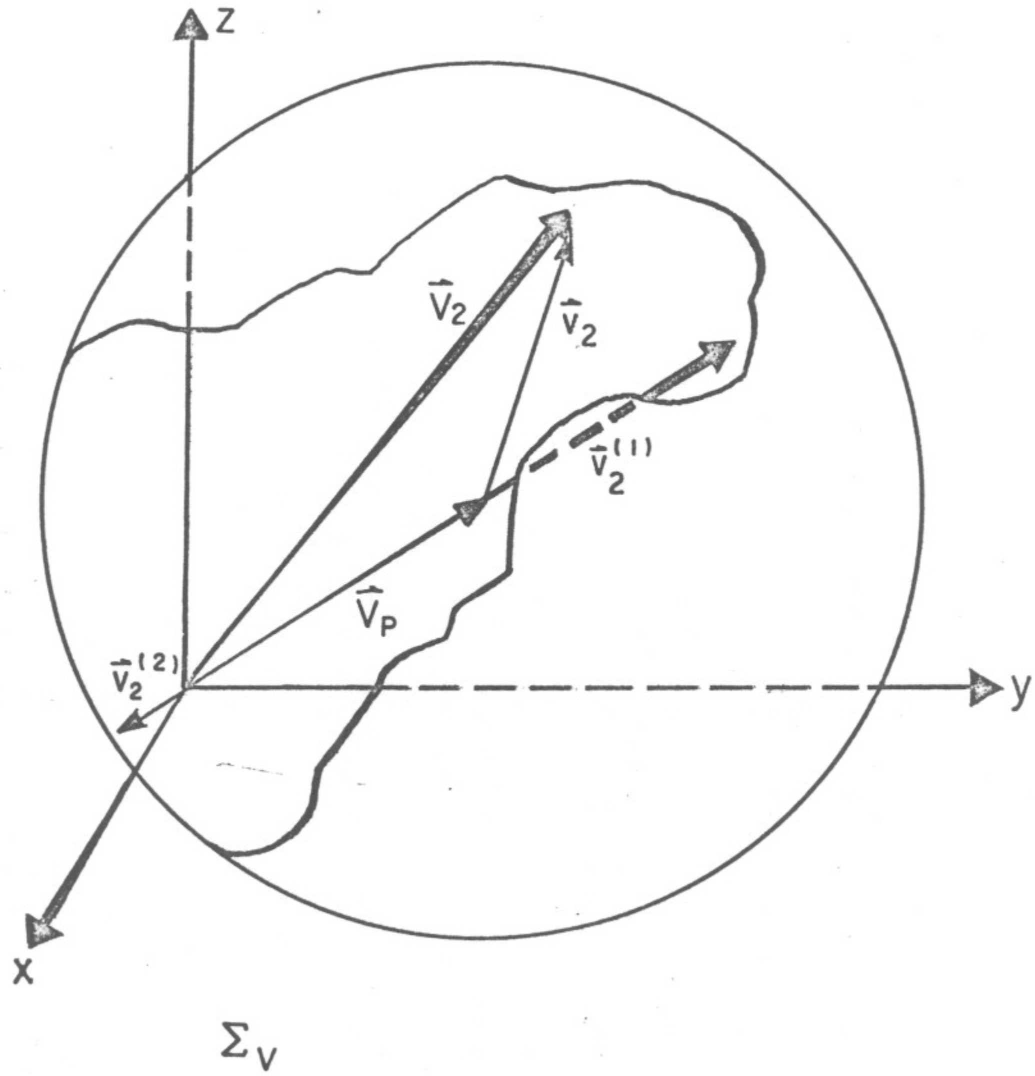


Figure 7. Velocity Surface (Hodosurface) of v_2 for all Possible Post-Encounter Trajectories Corresponding to a given P_o-P_1 Transfer when $v > V_P$.

to obtain two different solar impact trajectories. They can be easily calculated by substituting $\vec{V}_2 = \pm \hat{R}_p V_2$ into the energy exchange equation (2.4.5) and solving for V_2. For $\vec{V}_2 = - \hat{R}_p V_2$ we obtain

$$V_2 = - (\vec{V}_p \cdot \hat{R}_p) \pm \left[(\vec{V}_p \cdot \hat{R}_p)^2 - 2\vec{V}_p \cdot \vec{V}_1 + V_1^2\right]^{1/2}$$

$$= - (\vec{V}_p \cdot \hat{R}_p) \pm \left[(\vec{V}_p \cdot \hat{R}_p)^2 + v^2 - V_p^2\right]^{1/2} .$$

Hence, since we require $V_2 > 0$ and $v > V_p$, the correct solution is

$$\vec{V}_2 = - \hat{R}_p \left\{ -(\vec{V}_p \cdot \hat{R}_p) + \left[(\vec{V}_p \cdot \hat{R}_p)^2 + v^2 - V_p^2\right]^{1/2} \right\}. \qquad (3.2.1)$$

Similarly, by setting $V_2 = + \hat{R}_p V_2$, we obtain

$$\vec{V}_2 = + \hat{R}_p \left\{ +(\vec{V}_p \cdot \hat{R}_p) + \left[(\vec{V}_p \cdot \hat{R}_p)^2 + v^2 - V_p^2\right]^{1/2} \right\} . \qquad (3.2.2)$$

The required planetary approach trajectories which will generate these post-encounter velocities can be calculated by the method given in section 3.1, using equations (3.1.9) through (3.1.14). The post-encounter trajectory generated by (3.2.1) will take the vehicle directly into the sun along a straight line. On the other hand, the post-encounter trajectory generated by (3.2.2) will take the vehicle on a straight line away from the Sun until it reaches a certain maximum distance. It will then fall back along the same straight line directly into the Sun. The post-encounter trajectories in both cases will have eccentricities equal to 1.

If $v < V_p$, a solar impact post-encounter trajectory with eccentricity 1 will not be possible. It is interesting, therefore, to consider the following question: Will the minimum energy post-encounter trajectory generated by the $\vec{V}_2$ given in (3.1.7) also be identical to the trajectory which will pass closest to the Sun (i.e., have minimum perihelion distance)? The answer to this question turns out to be true only under certain conditions.

Let q denote the post-encounter trajectory's perihelion distance. Then

$$q = a(1 - e) , \tag{3.2.3}$$

where we assume that the trajectory is elliptical. Since we will be concerned exclusively with the vehicle's post-encounter trajectory, the subscripts will be omitted from the various orbital parameters. We shall now determine the vector $\vec{V}_2$, which will extremize this function. (As soon as a required $\vec{V}_2$ is determined, the corresponding planetary approach trajectory required to generate this departing velocity can be obtained by (3.1.9)-(3.1.14).) As in the previous section, the extremization of a certain function can be accomplished by the Lagrange multiplier method with $F_1 = 0$ acting as a constraining equation. Hence, the extremals of q will be the solutions to the following system of equations:

$$\frac{\partial q}{\partial x} - \lambda_1 \frac{\partial F_1}{\partial x} = 0$$

$$\frac{\partial q}{\partial y} - \lambda_1 \frac{\partial F_1}{\partial y} = 0$$

$$\frac{\partial q}{\partial z} - \lambda_1 \frac{\partial F_1}{\partial z} = 0$$

$$F_1 = 0.$$

The first three equations become

$$\left.\begin{aligned}\frac{\partial a}{\partial x}(1-e) - a\frac{\partial e}{\partial x} - 2\lambda_1(x-u_1) &= 0\\ \frac{\partial a}{\partial y}(1-e) - a\frac{\partial e}{\partial y} - 2\lambda_1(y-u_2) &= 0\\ \frac{\partial a}{\partial z}(1-e) - a\frac{\partial e}{\partial z} - 2\lambda_1(z-u_3) &= 0\end{aligned}\right\} \tag{3.2.4}$$

By making use of the fact that

$$e^2 = 1 - \frac{\mu_s}{ah^2}$$

and

$$h^2 = \frac{R^2V^2 - (\vec{R}\cdot\vec{V})^2}{\ell^2} ,$$

it follows that

$$\frac{\partial e}{\partial x} = \frac{\ell}{2a^2e}\frac{\partial a}{\partial x} - \frac{1}{ae\mu_s}\left[R^2x - R_1(\vec{R}\cdot\vec{V})\right]$$

$$\frac{\partial e}{\partial y} = \frac{\ell}{2a^2e}\frac{\partial a}{\partial y} - \frac{1}{ae\mu_s}\left[R^2y - R_2(\vec{R}\cdot\vec{V})\right]$$

$$\frac{\partial e}{\partial z} = \frac{\ell}{2a^2e}\frac{\partial a}{\partial z} - \frac{1}{ae\mu_s}\left[R^2z - R_3(\vec{R}\cdot\vec{V})\right].$$

Now, from the orbit energy equation (1.1.23), one finds

$$\frac{\partial a}{\partial x} = \frac{2a^2}{\mu_s}x$$

$$\frac{\partial a}{\partial y} = \frac{2a^2}{\mu_s}y$$

$$\frac{\partial a}{\partial z} = \frac{2a^2}{\mu_s}z .$$

When these results are substituted into (3.2.4), we obtain

$$\alpha_1\vec{R}_p + \alpha_2\vec{V}_2 = \vec{V}_p , \tag{3.2.5}$$

where
$$\alpha_1 = \frac{\vec{R}_p \cdot \vec{V}_2}{2\lambda_1 \mu_s e} \tag{3.2.6}$$

$$\alpha_2 = 1 - \frac{R_p^2 - (1-e)^2 a^2}{2\lambda_1 \mu_s e} \tag{3.2.7}$$

Hence, in view of (3.2.5), we obtain

$$(\vec{V}_p - \alpha_1 \vec{R}_p) \times \vec{V}_2 = 0 \ . \tag{3.2.8}$$

If, instead of extremizing the post-encounter trajectory's perihelion distance q, we extremize its aphelion distance $Q = a(1+e)$ (where we assume that all post-encounter trajectories are elliptical), we could repeat the above calculations and find that equation (3.2.5) would become

$$\beta_1 \vec{R}_p + \beta_2 \vec{V}_2 = \vec{V}_p \quad , \tag{3.2.9}$$

where

$$\beta_1 = - \frac{\vec{R}_p \cdot \vec{V}_2}{2\lambda_1 \mu_s e} = - \alpha_1 \tag{3.2.10}$$

$$\beta_2 = 1 + \frac{R_p^2 - (1+e)^2 a^2}{2\lambda_1 \mu_s e} \quad . \tag{3.2.11}$$

Hence, it follows from (3.2.5) and (3.2.9) that the post-encounter trajectory which will either minimize its perihelion distance or maximize its aphelion distance must lie in the orbital plane of P_1 . This is a very interesting result since the pre-encounter transfer trajectory $P_o - P_1$ may have any arbitrary inclination relative to P_1's orbital plane.

Suppose we assume that, at encounter, P_1 is either at its point of aphelion or point of perihelion in its orbit about the Sun. Then $\vec{V}_p \perp \vec{R}_p$. In this situation, let $\vec{V}_2$ be chosen to be the departing velocity vectory which minimizes the vehicle's post-encounter orbital energy. Hence, $\vec{V}_2$ is defined by (3.1.7). In this case $\alpha_1 = 0$, and (3.2.8) is satisfied. This suggests that these conditions might also satisfy (3.2.5), in which case the trajectory would also minimize its perihelion distance. Since (3.2.8) does not imply (3.2.5), it is necessary to perform some calculations. First, it is clear that the post-encounter trajectory will be at its aphelion point immediately after encounter (i.e., $\vec{R}_p$ will correspond to the post-encounter trajectory's aphelion point.) Hence, $R_p = a(1+e)$. Also, in view of the energy equation, the post-encounter trajectory's semi-major axis is determined:

$$a = \frac{\mu_s R_p}{2\mu_s - R_p V_2^2} ,$$

where $V_2 = V_p - v$. Consequently, it follows from (3.2.5) that

$$\left[1 - \alpha_2\left(1 - \frac{v}{V_p}\right)\right]\vec{V}_p = 0.$$

Hence, if (3.2.5) is to be satisfied, we must determine if it is possible to choose λ_1 such that

$$1 - \alpha_2\left(1 - \frac{v}{V_p}\right) = 0.$$

By making use of (3.2.7), the equation becomes

$$\frac{R_p^2 - (1-e)^2 a^2}{2\lambda_1 \mu_s e} + \left[1 - \frac{R_p^2 - (1-e)^2 a^2}{2\lambda_1 \mu_s e}\right]\frac{v}{V_p} = 0 .$$

This equation can be written as

$$R_p^2 - (1-e)^2 a^2 + 2\lambda_1 \mu_s e \left(\frac{\frac{v}{V_p}}{1 - \frac{v}{V_p}} \right) = 0 \ .$$

However, since

$$R_p = a(1+e) \ ,$$

we obtain

$$2 + \lambda_1 \frac{\mu_s}{a^2} \left(\frac{\frac{v}{V_p}}{1 - \frac{v}{V_p}} \right) = 0 \ .$$

Therefore, if

$$\lambda_1 = -2 \frac{a^2}{\mu_s} \left(\frac{1 - \frac{v}{V_p}}{\frac{v}{V_p}} \right) ,$$

equation (3.2.5) will be satisfied. These calculations can be repeated for the case when $\vec{V}_2$ is defined to be that departing velocity vector given by (3.1.6) which maximizes the post-encounter trajectory's orbital energy. In this situation, equation (3.2.9) will be satisfied. Consequently, we obtain the following interesting result: Suppose a free-fall interplanetary space vehicle encounters a planet P_1 with relative speed $v < V_p$ when the planet is at its aphelion or perihelion points. Then the vehicle's post-encounter trajectory which minimizes its orbital energy will also minimize its perihelion distance. Similarly, the trajectory which maximizes its orbital energy will also maximize its aphelion distance. If the planet is not at its aphelion or perihelion points when encounter occurs, then it will be impossible to extremize these properties simultaneously with one post-encounter trajectory in either case.

Except for Pluto, all planets move about the Sun in very nearly circular orbits. Hence, $\vec{R}_p \cdot \vec{V}_p \approx 0$ will almost always be true in practical situations. Therefore, approach trajectories which minimize energy will also minimize perihelion distance. Those that maximize energy will also maximize aphelion distance.

If the gravitational fields of Mars or Venus are used to deflect an instrumented probe on trajectories generated by (3.2.5) or (3.2.9), the resulting distances of closest approach will almost always be negative. Therefore, as in the previous section, it will be necessary to consider the problem when the distance of closest approach to the surface of P_1 is specified as part of the initial conditions $(P_o,t_o,P_1,t_1;d)$. Proceeding according to the method of Lagrange, the extremals of q will be found by solving the following system of equations:

$$\frac{\partial q}{\partial x} - \lambda_1 \frac{\partial F_1}{\partial x} - \lambda_2 \frac{\partial F_2}{\partial x} = 0$$

$$\frac{\partial q}{\partial y} - \lambda_1 \frac{\partial F_1}{\partial y} - \lambda_2 \frac{\partial F_2}{\partial y} = 0$$

$$\frac{\partial q}{\partial z} - \lambda_1 \frac{\partial F_1}{\partial z} - \lambda_2 \frac{\partial F_2}{\partial z} = 0$$

$$F_1 = 0$$

$$F_2 = 0 \ ,$$

where F_1 and F_2 are given by (3.1.1) and (3.1.17) respectively. In view of our previous calculations (i.e., equations (3.2.3) through (3.2.7)), it is obvious that the first three of the above equations can be expressed as

$$\alpha_1\vec{R}_p + \alpha_2\vec{V}_2 + \alpha_3\vec{v}_1 = \vec{V}_p \quad , \qquad (3.2.12)$$

where α_1 and α_2 are given by (3.2.6) and (3.2.7) respectively, and where $\alpha_3 = -\lambda_2$. This equation implies that, in general, the post-encounter trajectory will not be in the orbital plane of P_1 .

Finding the extremal values of q and Q subject to the two constraining equations $F_1 = 0$ and $F_2 = 0$ is most conveniently accomplished by using the constraining equations to express y and z as functions of x and solving the equations

$$\frac{dq}{dx} = 0 \qquad\qquad \frac{dQ}{dx} = 0 \ .$$

Although the resulting functions $\frac{dq}{dx}$ and $\frac{dQ}{dx}$ will have complicated forms, the equations can be easily solved with the aid of digital computers. The functions $y(x)$ and $z(x)$ are given by

$$z(x) = \frac{\sigma_2 \pm \sqrt{\sigma_2^2 - \sigma_1\sigma_3}}{\sigma_1}$$

$$y(x) = \frac{\gamma - v_1 x - z(x)}{v_2} \ ,$$

where

$$\sigma_1 = 1 + \left(\frac{v_3}{v_2}\right)^2$$

$$\sigma_2 = \left\{u_2 - \left[\frac{\gamma - v_1 x}{v_2}\right]\right\}\frac{v_3}{v_2} - u_3$$

$$\sigma_3 = (x-2u_1)x + \left[\frac{\gamma-v_1x}{v_2} - 2u_2\right]\left(\frac{\gamma-v_1x}{v_2}\right) + \sigma .$$

Since $V_2 \leqq v + V_p$ the domain of x will be contained in the interval $[-(v + V_p), (v + V_p)]$.

Before closing this section I shall consider the following problem: Suppose P_o and P_1 have circular co-planar orbits. Suppose also that $P_o - P_1$ is a Hohmann transfer. (A Hohmann transfer is one which sweeps out 180^o about the Sun and is tangent to both the launch planet's orbit and the encountered planet's orbit.) What is the launch trajectory's absolute minimum hyperbolic excess velocity v_o at P_o such that, after encountering P_1 , the vehicle will escape the entire solar system? (The distance of closest approach is to be disregarded.) It is interesting to note that the radius of P_1's orbit is not given; hence, this must also be determined.

First of all we shall use the fact that a minimum energy escape trajectory is parabolic. Now, in view of (1.1.23),

$$a = \frac{R_p\mu_s}{2\mu_s - R_pV_2^2} .$$

Therefore, if the post-encounter trajectory is parabolic, $a = \infty$ or

$$2\mu_s - R_pV_2^2 = 0 . \tag{3.2.13}$$

We have seen in the previous section that a $\vec{V}_2$ given by (3.1.6) will give the probe a maximum energy post-encounter trajectory. In this case, $V_2 = v_1 + V_p$. Hence,

$$2\mu_s = R_p(v_1 + V_p)^2 \ .$$

Since $\Delta V = v\,[2(1 - \cos\Phi)]^{1/2}$ (where $\Phi = \sphericalangle\, \vec{V}_p, \vec{v}_1$) then to obtain maximum ΔV from P_1, Φ should be 180°. This means that the orbit of P_1 must be outside the orbit of P_o. (See figure 8). Consequently, $v_1 = V_p - V_1$; and we obtain

$$\sqrt{2\mu_s} = \sqrt{R_p}(2V_p - V_1) \qquad (3.2.14)$$

Since $P_o - P_1$ is a Hohmann transfer, its semi-major axis a_{01} is given by

$$2a_{01} = R_p + R_o \ .$$

Also, since P_1 has a circular orbit, P_1's semi-major axis a_p must be equal to R_p. Consequently, by making use of the energy equation (1.1.23), equation (3.2.14) becomes

$$\sqrt{2\mu_s} = \sqrt{R_p}\left\{2\left(\frac{\mu_s}{R_p}\right)^{1/2} - \left[\mu_s\left(\frac{2}{R_p} - \frac{2}{R_o+R_p}\right)\right]^{1/2}\right\}.$$

This reduces to

$$1 = \sqrt{2} - \left(\frac{R_o}{R_o+R_p}\right)^{1/2} .$$

The solution is

$$R_p = 2(1 + \sqrt{2})R_o \ .$$

Hence the minimum launch hyperbolic excess velocity is

$$v_o = V_o - V(P_o) = \left(\frac{\mu_s}{R_o}\right)^{1/2}\left\{2(\sqrt{2} - 1)^{1/2} - 1\right\}.$$

For the case where P_o = Earth (R_o = 1 A.U.), this minimum required hyperbolic excess velocity is 8.58 km/sec. The

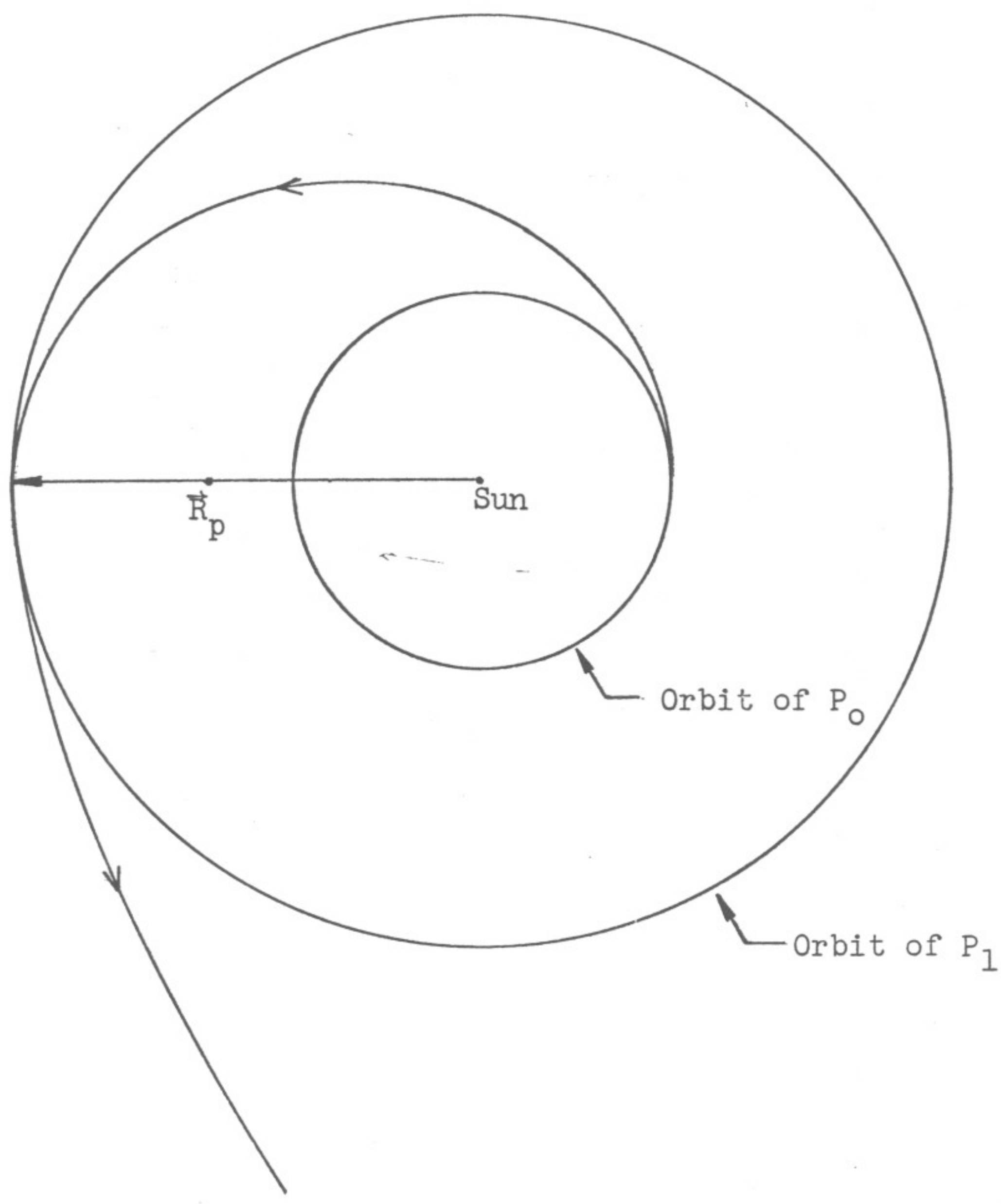


Figure 8. Hohmann transfer to P_1 resulting in parabolic post-encounter trajectory

radius of P_1's orbit is 4.83 A.U. The semi-major axis of Jupiter's orbit is 5.2 A.U. Its orbit is almost circular and coplanar with the Earth's. Hence, the calculations suggest that it may be possible to use minimum energy transfer trajectories to Jupiter to achieve high energy post-encounter trajectories which escape the entire solar system. Detailed numerical calculations indicate that this is in fact true. In order to obtain a parabolic solar system escape trajectory via a direct launch from Earth, the required launch hyperbolic excess is 12.43 km/sec.

3.3 Approach Trajectories which Maximize Post-Encounter Orbital Planes of Inclination

In the previous two sections we have determined Gravity Thrust trajectories designed for unmanned instrumented probes of interplanetary regions close to the ecliptic plane. In order to obtain a complete three-dimensional environmental study of our entire solar system, it will be necessary to explore regions of space far above and below this plane. The classical "brute force" direct ascent transfer trajectories into these regions will require very high launch hyperbolic excess velocities v_o. These missions, however, can be accomplished using Gravity Thrust trajectories with only a small fraction of the direct transfer energies.

Referring back to figure 7, we notice that, if $v_1 > V_p$, it is possible to give $\vec{V}_2$ any desired direction. In this situation, it is possible to obtain post-encounter trajectories which have their orbital planes precisely perpendicular to the ecliptic plane.

Let $\hat{K}$ denote a unit vector pointing above the ecliptic plane and normal to it. Let $\hat{R}$ be a second unit vector defined by

$$\hat{R} = \frac{\vec{R}_p \times (\hat{K} \times \vec{R}_p)}{|\vec{R}_p \times (\hat{K} \times \vec{R}_p)|} .$$

These vectors are illustrated in figure 9. If the departing velocity vector $\vec{V}_2$, defined by

$$\vec{V}_2 = V_2 \hat{R} ,$$

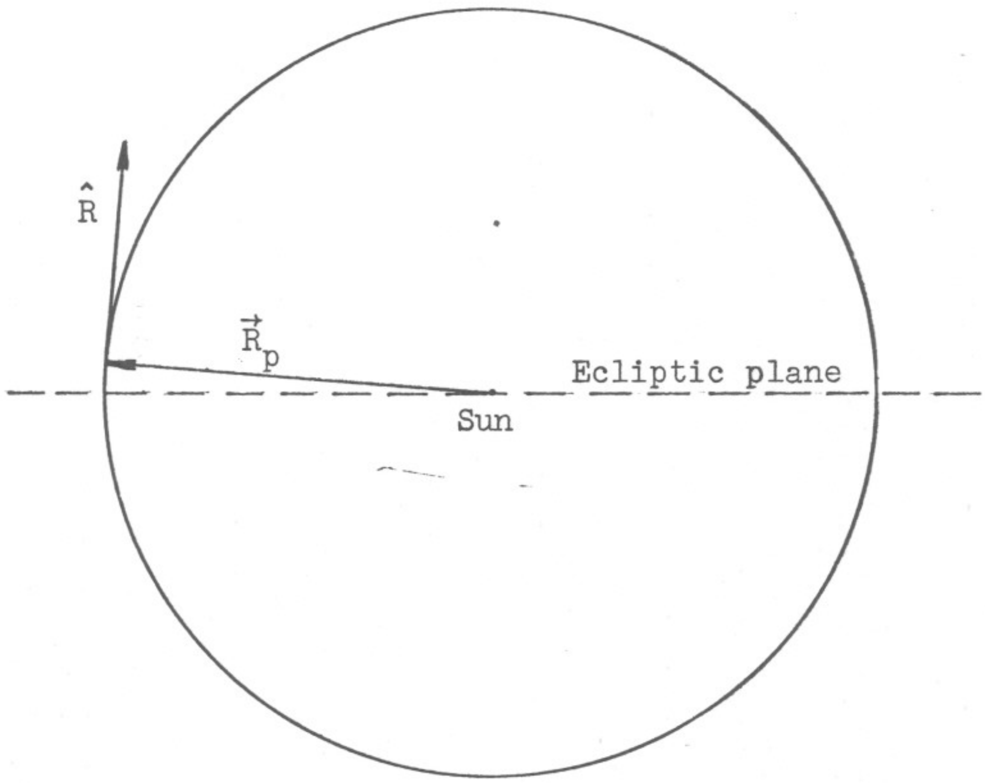


Figure 9. Post-encounter 90° out-of-ecliptic orbit

is substituted into the energy exchange equation, we obtain

$$V_2 = \vec{V}_p \cdot \hat{R} \pm \left[(\vec{V}_p \cdot \hat{R})^2 - (v^2 - V_p^2) \right]^{1/2} .$$

Since $V_2 > 0$ the equation gives

$$V_2 = \vec{V}_p \cdot \hat{R} + \left[(\vec{V}_p \cdot \hat{R})^2 + (v^2 - V^2) \right]^{1/2} .$$

In a similar manner, if we define

$$\vec{V}_2 = - V_2 \hat{R}$$

we obtain

$$V_2 = -(\vec{V}_p \cdot \hat{R}) + \left[(\vec{V}_p \cdot \hat{R})^2 + (v_1^2 - V_p^2) \right]^{1/2} .$$

Hence, the two possible departing velocities $\vec{V}_2$ which will generate post-encounter orbits normal to the ecliptic plane are

$$\vec{V}_2 = \left\{ (\vec{V}_p \cdot \hat{R}) + \left[(\vec{V}_p \cdot \hat{R})^2 + (v_1^2 - V_p^2) \right]^{1/2} \right\} \hat{R} \qquad (3.3.1)$$

and

$$\vec{V}_2 = \left\{ -(\vec{V}_p \cdot \hat{R}) + \left[(\vec{V}_p \cdot \hat{R})^2 + (v_1^2 - V_p^2) \right]^{1/2} \right\} \hat{R} . \qquad (3.3.2)$$

If $\vec{V}_p \cdot \vec{R} > 0$, then the post-encounter trajectory generated by (3.3.1) will, in general, reach a greater distance from the ecliptic plane than the trajectory generated by (3.3.2). On the other hand, if $\vec{V}_p \cdot \hat{R} < 0$, then the reverse will be true. The required planetary approach trajectories corresponding to a given initial P_o-P_1 transfer which will generate these post-encounter trajectories can be easily computed by equations (3.1.9) through (3.1.14). These Gravity Thrust trajectory profiles can be obtained by using the gravitational field of Jupiter. The required distances of closest approach associated with relatively low energy Earth-Jupiter transfers are sufficiently great so as to avoid any possible atmospheric drag at encounter.

Unfortunately, if the fields of Venus or Mars are used to obtain high inclination post-encounter trajectories, the required distances of closest approach will be negative. This difficulty can be resolved by giving the distance of closest approach a preassigned value. Hence, our problem becomes one of extremizing the post-encounter trajectory's orbital inclination i, subject to the given initial conditions $(P_o, P_1, t_o, t_1; d)$. Let Σ denote a heliocentric ecliptic coordinate system. Then

$$\cos i = \frac{h_3}{h} .$$

Hence, the problem of maximizing i is equivalent to the problem of minimizing the function

$$C = \frac{h_3}{h} .$$

This can be carried out most conveniently by using the two equations of constraint $F_1 = 0$ and $F_2 = 0$ as a means for expressing y and z as functions of x. Hence, the problem reduces to that of finding the solutions of

$$\frac{dC}{dx} = 0 . \tag{3.3.3}$$

Although the function $\frac{dC}{dx}$ takes on a rather complicated form, the solutions to (3.3.3) can be easily obtained with the aid of a digital computer.